30

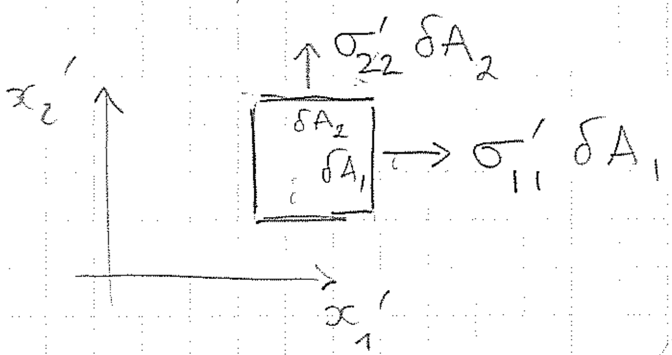
HW.1. σ is a symmetric m-x $\Rightarrow$ can be diagonalized by an orthogonal transform $\rightarrow$

$\rightarrow$ i.e. given σ^{ij} in some coordinate frame, we can find always another one s.t

$$\sigma' = O \sigma O^T \text{ is diagonal}$$

i.e.
$$\sigma^{i'j'} = \begin{pmatrix} \sigma^{1'1'} & & 0 \\ 0 & \sigma^{2'2'} & \\ 0 & 0 & \sigma^{3'3'} \end{pmatrix}$$

in such a frame we have only NORMAL STRESSES



NOTE: $\sigma^{ij}(\vec{x}, t)$

so a frame diagonalizing σ^{ij} at some $\vec{x}$ won't accomplish this at another $\vec{x}$ —

— don't take this as saying that stresses in a general fluid are always ONLY NORMAL —

— couldn't be further from truth!!

* true at every point locally

* ANOTHER PROPERTY OF MATRICES UNDER ORTHOGONAL TRANSFORMS IS THAT THEIR TRACES ARE INVARIANT:

$$\mathrm{tr}\,\sigma = \sigma^{11} + \sigma^{22} + \sigma^{33}$$

$$= \Sigma \text{ of diagonal elements}$$

$\mathrm{tr}\,\sigma = \mathrm{tr}\,\mathcal{O}^T\sigma'\mathcal{O} = \mathrm{tr}\,\mathcal{O}\,\mathcal{O}^T\sigma' = \mathrm{tr}\,\sigma'$

express the σ in another frame

cyclicity of trace

let's then write σ as follows:

$$\sigma^{ij} \equiv \underbrace{\left(\delta^{ij}\tfrac{1}{3}\sigma^{kk}\right)} + \underbrace{\left(\sigma^{ij} - \tfrac{1}{3}\delta^{ij}\sigma^{kk}\right)}$$

$$\left(\mathrm{tr}\,\sigma \equiv \sigma^{kk} \equiv \sum_{k=1}^{3}\sigma^{kk} = \sigma^{11}+\sigma^{22}+\sigma^{33}\right)$$

clearly this is an identity as I added & subtracted

only diagonal stresses, same in each direction

"traceless part" of σ

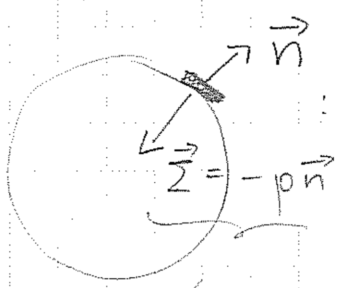
Consider now a small element (e.g. a sphere) of a fluid. Let's choose an orientation of axes such that σ is diagonal (always possible) at the location of the sphere $\longrightarrow$

so we have

$$\|\sigma\| = \begin{pmatrix} \frac{1}{3}\operatorname{tr}\sigma & 0 & 0 \\ 0 & \frac{1}{3}\operatorname{tr}\sigma & 0 \\ 0 & 0 & \frac{1}{3}\operatorname{tr}\sigma \end{pmatrix} + \underbrace{\begin{pmatrix} \sigma_{xx} - \frac{1}{3}\operatorname{tr}\sigma & & \\ & \sigma_{yy} - \frac{1}{3}\operatorname{tr}\sigma & \\ & & \sigma_{zz} - \frac{1}{3}\operatorname{tr}\sigma \end{pmatrix}}_{\text{the sum of these 3 is zero, so write as}}$$

let's call $\frac{1}{3}\operatorname{tr}\sigma = \underbrace{-p}_{\text{just a name for now}}$

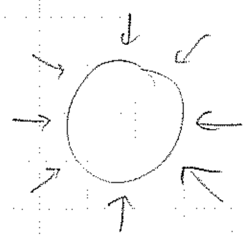
$$\|\sigma\| = \underbrace{\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}}_{\text{this part is uniform } (\sim \delta_{ij})} + \underbrace{\begin{pmatrix} \tilde{\sigma}_{xx} & 0 & 0 \\ 0 & \tilde{\sigma}_{yy} & 0 \\ 0 & 0 & -(\tilde{\sigma}_{xx} + \tilde{\sigma}_{yy}) \end{pmatrix}}_{(**)}$$



: recall $\Sigma^i(n) = \underbrace{\sigma^{ij}}\, n^j$, from 1st term

$$\Sigma^i(\vec{n}) = -p\,\delta^{ij} n^j$$

$$\vec{\Sigma}(\vec{n}) = -p\,\vec{n}$$

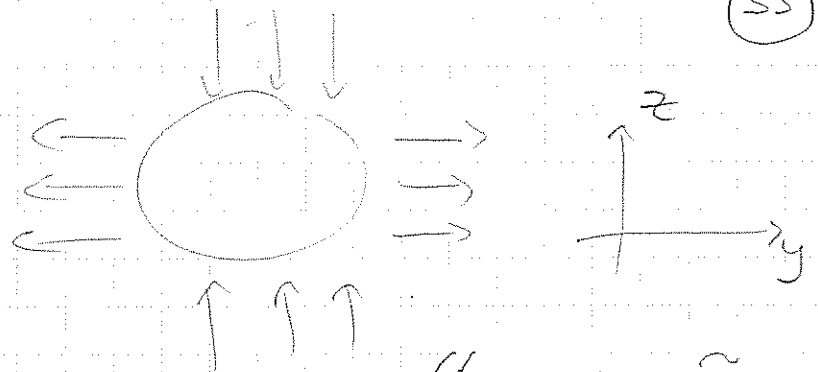


so 1st term: • force on surface element is $\vec{\Sigma}\, d^2S(\vec{n})$ and sphere is isotropically compressed

• force per unit area is always $\perp$ & isotropic

2nd term? as (**) shows, compression in some direction and expansion in other (since trace $= 0$)

so we have, e.g.:

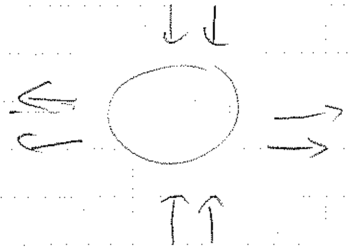


((assuming $\tilde{\sigma}_{xx} = 0$, compression in z same as tension in y shown for $\sigma_{yy} > 0$))

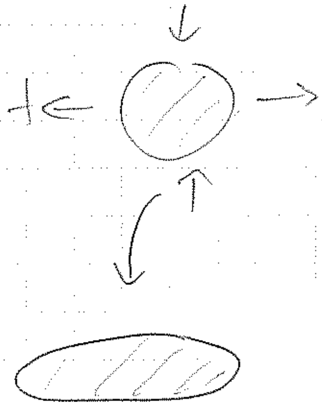
Now, recall that we argued that a FLUID can not withstand deformation.

Thus, for a fluid at rest,

since

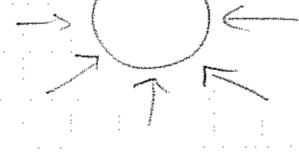


would tend to distort



it must be that 2nd term in σ (traceless) $\equiv 0$
(otherwise would move)

1st term – uniform compression $\Longleftrightarrow$



does not change shape –

– so for a fluid at rest we can only have

$$\sigma_{ij} = -p\,\delta_{ij} \; ; \quad p = \text{static fluid pressure}$$

$\boxed{\text{SCALAR}}$

(force / unit area)

moral: for a fluid at rest we argued that

$\sigma_{ij} = -p\,\delta_{ij}$ $\Longleftrightarrow$ NOTE THIS IS TRUE IN ANY FRAME

(EVEN THOUGH we used local diagonalization at a given point to get to it)

stress tensor for a fluid at rest is that of isotropic compression.

If we go to our equ of ρ (26)

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial x^j}(\rho u^j)$$

$$\frac{\partial}{\partial t}(\rho u^j) = \rho F^i - \frac{\partial}{\partial x^j}(\rho u^i u^j - \sigma^{ij})$$

& specialize to a fluid at rest $\rightarrow u^i \equiv 0$

$\rightarrow \dot{\rho} = 0$

$\rightarrow \sigma^{ij} = -p\,\delta^{ij}$

we find that 1st equ is $\equiv 0$,

while 2nd gives $0 = \rho F^i - \frac{\partial}{\partial x^i} p$

$$\boxed{\rho F^i = \frac{\partial}{\partial x^i} p.}$$

$\Longleftarrow$ equilibrium condition (for fluid known from elementary physics!)

Example: imagine we have a gravity force as the only volume force, force per unit mass is $F^i \equiv -\frac{\partial}{\partial x^i}\phi(\bar{x})$

gravitational potential

Then we have

$$\rho F^i = \frac{\partial}{\partial x^i} p$$

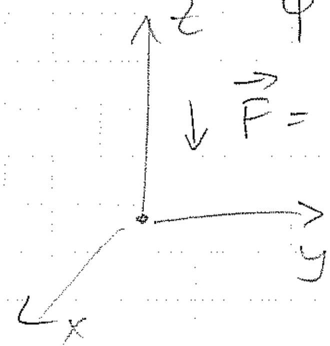
F^z only $\neq 0$ component,

hence $p = p(z)$ only

& $$\underbrace{\frac{d p(z)}{dz} = -g\rho(z)}$$

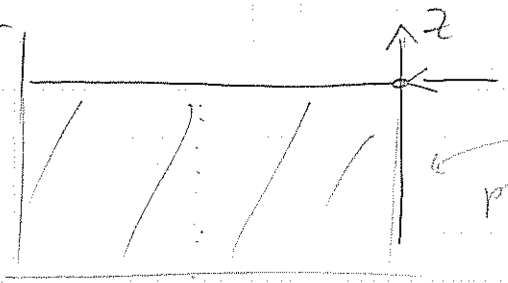
$\phi = gz$

$\downarrow \vec{F} = -\vec{\nabla}\phi = -g\vec{e}_z$



▪ if density $\rho(z)$ is uniform, $p(z) = p(0) - g\rho z$

e.g. water



pressure increases

▪ if, instead (atmosphere) we have approx. T=const

$pV = NkT \longrightarrow$ ideal gas $\rightarrow \frac{m\, p}{m\left(\frac{N}{V}\right)} = kT \approx$ const

(m - mass of air molecule (N_2))

$\rightarrow \frac{p}{\rho} \approx$ const, then $\frac{dp}{dz} = -\frac{gmp}{kT} \rightarrow p = p(0)\, e^{-\frac{mgz}{kT}}$

moral (A): our fancy general equations are sensible (give elementary physics results)

moral (B): even in these simple cases, some information outside FM is needed to find closed form solutions (i.e. water has ≈ constant density while atmosphere has ≈ constant T)

We now go back to study the general conservation equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho = -\dfrac{\partial}{\partial x^j}(\rho u^j) \\[2ex] \dfrac{\partial}{\partial t}(\rho u^i) = \rho F^i - \dfrac{\partial}{\partial x^j}(\rho u^i u^j - \sigma^{ij}) \end{cases}$$

let's define $\sigma^{ij} = -p\delta^{ij} + \hat{\sigma}^{ij}$

no loss of generality here!

"stress tensor"

for now, just a name

"viscous stress tensor"

If $\hat{\sigma} = 0$ we have

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -\dfrac{\partial}{\partial x^j}(\rho u^j) \\[2ex] \dfrac{\partial}{\partial t}(\rho u^i) = \rho F^i - \dfrac{\partial}{\partial x^j}(\rho u^i u^j + p\delta^{ij}) \end{cases}$$

& $\sigma^{ij} = -p\delta^{ij}$

aka "ideal fluid"

→ no viscosity, $\hat{\sigma} = 0$

Let's dwell on the "ideal" fluid equations for a bit — before delving into viscosity & dissipation:

use continuity (mass conserv.) equ to plug into 2nd (momentum conservation) to get

$$\dot{\rho} u^i + \rho \dot{u}^i = \rho F^i - \frac{\partial}{\partial x^j}(\rho u^j) u^i - \rho u^j \frac{\partial u^i}{\partial x^j} - \frac{\partial}{\partial x^i} p$$

$\downarrow$ continuity

$$-\cancel{\frac{\partial}{\partial x^j}(\rho u^j)}\, u^i + \rho \dot{u}^i = \rho F^i - \cancel{\frac{\partial}{\partial x^j}(\rho u^j)}\, u^i - \rho u^j \frac{\partial u^i}{\partial x^j} - \frac{\partial p}{\partial x^i}$$

$$\rho \frac{D u^i}{D t} = \rho F^i - \frac{\partial p}{\partial x^i}$$

$$\boxed{\frac{D u^i}{D t} = F^i - \frac{1}{\rho} \frac{dp}{dx^i}}$$

or: $\dot{\vec{u}} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \vec{F} - \frac{1}{\rho} \vec{\nabla} p$

▷ this is known as Euler's equation (1755).

- we could have very simply arrived at it by assuming that the only surface force is due to the pressure of the fluid

- we've talked about mass- & momentum- conservation, but have not mentioned energy balance yet --- this is simplest for ideal fluid, where energy of a fluid element can only change due to work of external forces (body) or surface forces (pressure) — and there's **no** heat exchange involved.