So, look at the kinetic energy of a fixed (38)

V of fluid (<u>ideal</u> for now, $\sigma_{ij} = -p\delta_{ij}$)

$$\underbrace{\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\rho \vec{u}^2\right) dV}_{} = ?$$

$$\frac{\partial}{\partial t}\mathcal{E}_{kin}(V) =$$

<u>goal:</u> find r.h.s., using equations of motion of Euler (& continuity) & give them physical interpretation (as making up energy balance of ideal fluid)

Slowly now:

$$\partial_i := \frac{\partial}{\partial x^i}$$

V is fixed: $\frac{\partial}{\partial t}\int_V \frac{1}{2}\rho \vec{u}^2 dV =$

$$= \int_V \frac{1}{2}\dot{\rho}\,\vec{u}^2 dV + \int_V \rho\, \vec{u}\cdot\dot{\vec{u}}\, dV = \quad \text{(use continuity)}$$

$$= -\int_V \frac{1}{2}\partial_i(\rho u_i)\vec{u}^2 dV + \int_V \rho\,\vec{u}\left(\underset{\uparrow \text{ use Euler}}{\frac{D\vec{u}}{Dt}} - (u^j\partial^j)\vec{u}\right) dV$$

$$= -\int_V \frac{1}{2}\partial_i(\rho u_i)\vec{u}^2 dV + \int_V \rho\vec{u}\left(\vec{F} - \frac{1}{\rho}\vec{\nabla}p\right)dV - \int_V dV \rho \underbrace{u^i u^j \partial^j u^i}_{=\frac{1}{2}u^j\partial^j(\vec{u}^2)}$$

$$= -\int_V \frac{1}{2}\left[\partial_j(\rho u_j)\vec{u}^2 + (\rho u_j)\partial_j(\vec{u}^2)\right] dV$$
$$+ \int_V \rho\vec{u}\vec{F}\, dV - \int_V \vec{u}\cdot\vec{\nabla}p\; dV = \longrightarrow$$

$$= -\frac{1}{2}\int_V \partial_j (\rho u_j \vec{u}^2)\, dV + \int_V \rho \vec{u}\cdot\vec{F}\, dV$$

$$-\int_V \vec{u}\,\vec{\nabla}p\, dV = \quad \to \text{surface } \oint_{\partial V} u_j \frac{\rho\vec{u}^2}{2} d^2S^j$$

$$= -\underbrace{\oint_{\partial V} \frac{\rho\vec{u}^2}{2} u_j\, d^2S^j}_{\substack{\text{energy flux through} \\ \text{boundary of } V \\ (\text{sign: energy in } V \text{ decreases} \\ \text{of outward flux})}} + \underbrace{\int_V \vec{u}\cdot(\rho\vec{F})\, dV}_{\substack{\text{work of outside body forces in } V \\ \text{- if work} > 0, \text{ increase} \\ \text{energy inside } V}} - \underbrace{\int_V \vec{u}\cdot\vec{\nabla}p\, dV}_{**}$$

$$** = -\int_V \vec{u}\cdot\vec{\nabla}p\, dV = -\int_V \left(\vec{\nabla}(\vec{u}p) - p\,\vec{\nabla}\cdot\vec{u}\right) dV$$

$$= \int_V p\,\vec{\nabla}\cdot\vec{u}\, dV - \underbrace{\int_V \vec{\nabla}(\vec{u}p)\, dV}_{\oint_{\partial V = S} u^i(-p\, d^2S^i)}$$

$$\underbrace{u^i \underbrace{(-p\, d^2S^i)}_{\text{force on surface}}}_{\text{work of surface force}}$$

for ideal fluid

$$\frac{\partial}{\partial t}\mathcal{E}_{kin}(V) = -\underbrace{\oint_{\partial V} \frac{\rho\vec{u}^2}{2} u_j\, d^2S^j}_{\text{flux thru } S} + \underbrace{\int_V \vec{u}\rho\vec{F}\, dV + \oint_{\partial V} \vec{u}(-p)\, d^2\vec{S}}_{\text{work of body/surface forces/unit t.}}$$

$$+ \int_V p\,\vec{\nabla}\cdot\vec{u}\, dV$$

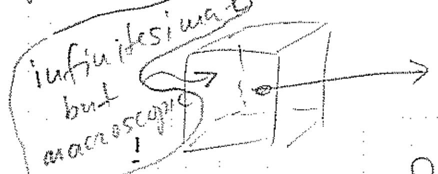
— what's $\int_V p \vec{\nabla}\cdot\vec{u}\, dV$?

(*) vanishes for incompressible fluid ($\vec{\nabla}\cdot\vec{u}=0$)

$\Longrightarrow$ so we understand energy balance in this case — only flux of energy in/out of V + work of pressure & body force. — clear!

(**) if fluid is compressible $\vec{\nabla}\cdot\vec{u} \neq 0$

Now, it is clear that if one compresses a gas w/out heat transfer its internal energy change — already mentioned example where $\frac{p}{\rho} = \frac{kT}{m}$, the ideal gas law. To interpret $\int_V p \vec{\nabla}\cdot\vec{u}\, dV$ term we need to include the internal energy per unit mass in the balance of energy equation:



$$\mathcal{E}_{int}(V) = \underbrace{\int_V \rho\, \varepsilon\, dV}_{\text{internal energy of gas in } dV} \qquad \underbrace{\frac{\rho \vec{u}^2}{2} dV}_{\text{kinetic energy}}$$

= TO BE CONTINUED =

Energy balance of an ideal compressible fluid

(⇒ IMPORTANT – THERMODYNAMIC INPUT NEEDED/REVIEWED ⇐)

- as already discussed, for an ideal fluid the only reason the internal energy of the fluid can change is work done on it due to compression / expansion by surface & body forces – no heat exchange & dissipation in fluid; in kinetic theory – – true local TD equil – no viscosi[ty] etc – (NO TRANSPORT-REGIME) (NONEQUIL) can only change due to work – – no heat
- apply 1st law of TD

$$dU = -p\,dV$$

– internal energy of a body of fixed # particles

if not fixed # particles – (– an extra term μdN needed –

– since # particles fixed ⇒ mass fixed (nonrelativistic)

So, for a fluid element of fixed mass M, we have

$$U = \varepsilon M, \text{ where } \varepsilon = \frac{\text{internal energy}}{\text{unit mass}}$$

if fluid has (mass) density ρ, then, of course $M = V\rho$

1st law of TD, properly applied, now says that $dU = M\, d\mathcal{E} = -p\, dV$

since fixed M

but $dV = d\left(\frac{M}{\rho}\right) = -\frac{M}{\rho^2}\, d\rho$

hence $M\, d\mathcal{E} = -p\, dV = M \frac{p}{\rho^2}\, d\rho$

or $\underbrace{d\mathcal{E} = \frac{p}{\rho^2}\, d\rho}_{\text{to make use of}}$ [and now we can forget about M – it was a trick to get this equ. properly].

this we need some more info – since depends on $\mathcal{E}, \rho, p \Longleftrightarrow$ usually we assume that the fluid is in local thermal equilibrium – in other words in the rest frame of the fluid (where it's at rest, also called "comoving frame") at any given point a small element of fluid obeys some appropriate equ. of state where two thermodynamic variables fix the rest:

i.e. $\longrightarrow (p, \rho), (p, T), (\rho, s) \ldots$

for ideal fluid it's very convenient to take mass density ρ & entropy per unit mass s as independent variables

i.e. $sM \equiv$ entropy of fluid of mass M

(NOT volume density, as # particles can change)

in a process w/out heat exchange, we'd have from 1st law:

$$dU = \underbrace{\delta Q}_{0\text{ - no heat}} - p\,dV$$

1st law more general for LATER: $d\varepsilon = T\,ds + \frac{p}{\rho^2}d\rho$

$$M d\varepsilon = T\,dS - p\,dV$$

$$M d\varepsilon = \underbrace{TM\,ds}_{\phi \;\rightarrow\; ds = 0} - p\,dV$$, but since $\delta Q = 0$

So s = const for ideal fluid, i.e. does not change in time for a physical element (FLUID)

So, we take ρ & s as parameters & have

$$p = p(\rho, s)$$
$$\varepsilon = \varepsilon(\rho, s)$$

D.I.Y.: What are these for an ideal gas? i.e. $p = p(\rho, s)$, $\varepsilon = \varepsilon(\rho, s)$

back to $\underbrace{d\varepsilon = \frac{p}{\rho^2}d\rho}$ ... We're interested in (meaning of d.)

N.B. AGAIN: LOCAL TH EQUIL RELATIONS APPLY TO PHYSICAL FLUID the change of the internal energy of the PHYSICAL FLUID ELEMENT IN TIME:

$$\frac{D\varepsilon}{Dt} = \frac{D\varepsilon(\rho,s)}{Dt} = \left(\frac{\partial\varepsilon(\rho,s)}{\partial\rho}\right)_s \frac{D\rho}{Dt} + \left(\frac{\partial\varepsilon(\rho,s)}{\partial s}\right)_\rho \underbrace{\frac{Ds}{Dt}}_{\substack{0 \\ \text{ideal}}}$$

but now for adiabatic $s = \text{const}$

$$d\varepsilon = \frac{p}{\rho^2}\,d\rho \rightarrow \left(\frac{\partial\varepsilon}{\partial\rho}\right)_s = \frac{p}{\rho^2}$$

hence $\displaystyle \frac{D\varepsilon}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt}$ ⟵ INPUT:

local TH. EQUIL.
+
1st LAW
+
IDEAL FLUID – NO HEAT

Where are we going?

We had on bottom of p. 39 / top of p. 40

$$\frac{d}{dt}\mathcal{E}_{kin}(V) =$$

$$= \underbrace{-\oint_{\partial V} d^2S^j\, u^j \frac{\rho\vec{u}^2}{2}}_{\text{energy flux through } \partial V} + \underbrace{\int_V \rho\,\vec{u}\cdot\vec{F}\,dV + \oint_{\partial V} -p\,\vec{u}\cdot d^2\vec{S}}_{\text{work of body \& surface forces on ideal fluid in } V}$$

$$+ \underbrace{\int_V p\,\vec{\nabla}\cdot\vec{u}\,dV}_{?}$$

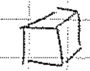
$= 0$ if incompressible

$\neq 0$ if compressible ⟹ must be related to $\frac{d}{dt}\mathcal{E}_{int}(V)$.

We're almost there ->

$$\begin{pmatrix}\text{internal energy of} \\ \text{fluid inside } V\end{pmatrix} \equiv E_{int}(V) = \int_V \underbrace{\rho\,\varepsilon\, dV}_{}$$

fixed volume in space & time

internal energy of fluid that happens to be in dV  at t

$$\frac{d}{dt} E_{int}(V) =$$

$$= \frac{d}{dt}\int_V \rho\,\varepsilon\, dV = \int_V (\rho\varepsilon)^{\circ}\, dV$$

$$= \int_V (\mathring{\rho}\,\varepsilon + \mathring{\varepsilon}\,\rho)\, dV =$$

but Now: $\dfrac{D\varepsilon}{Dt} = \dfrac{p}{\rho^2}\dfrac{D\rho}{Dt}$; thermodynamics of ideal fluid (local thermal equilibrium) assumption

$$\frac{\partial \varepsilon}{\partial t} + \vec{u}\vec{\nabla}\varepsilon = \frac{p}{\rho^2}\left(\mathring{\rho} + \vec{u}\vec{\nabla}\rho\right)$$

continuity

$$\frac{\partial \varepsilon}{\partial t} = -\vec{u}\vec{\nabla}\varepsilon + \frac{p}{\rho^2}\left(-\vec{\nabla}(\rho\vec{u}) + \vec{u}\cdot\vec{\nabla}\rho\right)$$

$$-\cancel{(\vec{\nabla}\rho)\vec{u}} - \rho\,\vec{\nabla}\vec{u} + \cancel{\vec{u}\cdot\vec{\nabla}\rho}$$

so $\mathring{\varepsilon} = -\vec{u}\cdot\vec{\nabla}\varepsilon - \dfrac{p}{\rho}\vec{\nabla}\cdot\vec{u}$

& knowing $\mathring{\rho} = -\vec{\nabla}(\rho\vec{u})$ from continuity

plug into →

$$\frac{d}{dt} E_{kin}(V) = \int_V (\dot{\rho}\, \varepsilon + \dot{\varepsilon}\, \rho)\, dV =$$

$$= \int_V \Big(\underbrace{-\big(\vec{\nabla}(\rho\vec{u})\big)\varepsilon - \rho\vec{u}\vec{\nabla}\varepsilon}_{} - p\vec{\nabla}\vec{u} \Big) dV =$$

$$-\vec{\nabla}(\varepsilon\rho\vec{u}) = -\varepsilon(\vec{\nabla}\rho\vec{u}) - \rho\vec{u}(\vec{\nabla}\varepsilon)$$

$$= \int_V \left(-\vec{\nabla}(\varepsilon\rho\vec{u})\, dV \right) - \int_V p\vec{\nabla}\vec{u}\, dV =$$

$$= -\oint_{\partial V = S} \varepsilon\rho\vec{u}\, d^2\vec{S} - \underbrace{\int_V p\vec{\nabla}\vec{u}\, dV}_{}$$

last term on p. 44

cancels the corresponding term in $\frac{d}{dt}\mathcal{E}_{kin}(V)$

Combine w/ $\frac{d}{dt}\left[\mathcal{E}_{kin}(V) + \mathcal{E}_{int}(V) \right] =$

$$= \frac{d}{dt} \int_V \left(\frac{\rho\vec{u}^2}{2} + \varepsilon\rho \right) dV = -\oint_{\partial V} d^2\vec{S}\, \vec{u} \left(\frac{\rho\vec{u}^2}{2} + \varepsilon\rho \right) dV$$

$$+ \oint_{\partial V} \left(-p\vec{u}\, d^2\vec{S} \right)$$

$$+ \int_V \rho\vec{u}.\vec{F}\, dV$$

General compressible

rate of change of <u>total</u> energy in V

||

flux of total energy thru ∂V + work of body or surface forces $\Longleftrightarrow$ IDEAL FLUID ENERGY BALANCE

So only thing left is to consider also dissipation & allow heat ——

& we'll have a very general set of equs.

* Note one important thing — role of thermodynamic considerations crucial.

Local thermal equilibrium assumed & corresponding thermodynamic laws should be applied in COMOVING FRAME — i.e. for physical fluid elements!

warning!

▷ if you're not careful can easily get crappy results (how do I know that, you think?)

Let's summarize what we've learned for an ideal liquid

continuity equ + mom-m conserv. ≡ continuity + Euler + equ. of state.

$$\left\{\begin{aligned} &\dot{\rho} = -\vec{\nabla}(\rho\vec{u}) \\ &(\rho u^i)^{\cdot} = \rho F^i - \partial_j(\rho u^j u^i + p\delta^{ij}) \\ &p = p(\rho, s) \end{aligned}\right\} \text{ equiv. to } \left\{\begin{aligned} &\dot{\rho} = -\vec{\nabla}(\rho\vec{u}) \\ &\dot{\vec{u}} + (\vec{u}\vec{\nabla})\vec{u} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p \\ &p = p(\rho, s) \end{aligned}\right.$$

4 unknowns — 4 equs $(\rho, \vec{u})$

$\Rightarrow$ if we assume s is same over entire liquid

(48)

⟹ ie. if you assume a uniform state of liquid @ initial time $s(\bar{x},t)\big|_{t=0}$ = const indep of $\bar{x}$,

then since $\frac{Ds}{Dt} = 0$ (entropy of physical volume element = const)

we'll have
$$\frac{\partial s(\bar{x},t)}{\partial t} = \underbrace{-\vec{u}\,\vec{\nabla}\, s(\vec{x},t)}$$

but if s is $\vec{x}$-indep at $t=0$, $\frac{\partial}{\partial t} s(\vec{x},t)\big|_{t=0} = 0$

& so it will never change (1st order equ!!)

→ so $s(x,t) = s_0 \;\; \forall \bar{x}, t$

for such a case.

then we only need $p = p(\rho)$

$$\boxed{\begin{array}{l} \text{for ideal gas } \; pV^{\gamma} = \text{const}, \;\; \gamma = \dfrac{f/2+1}{f/2} \qquad f=3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f=7 \\ \text{hence } \; \underline{p = \text{const}\, \rho^{+\gamma}, \; \gamma > 1} \qquad\qquad\qquad\qquad \text{etc} \end{array}}$$

→ (clearly not good for liquids — e.g. incompressible case where ρ = const)

Equs on rhs on bottom of p47 simplify in many cases — notably ※ stationary

$$\dot{\rho} = 0$$
$$\dot{\vec{u}} = 0$$

(because if ρ dep on t so will $\vec{u}$ from 2nd equ)

$$\vec{\nabla}(\rho\vec{u}) = 0$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p$$

Will come back to these

※ ~~stationary +~~ incompressible

$$\vec{\nabla}(\rho\vec{u}) = \underbrace{\vec{\nabla}\rho}\,\vec{u} + \rho\,\underbrace{\vec{\nabla}\vec{u}}_{0} = 0$$

$$\Rightarrow \rho = \text{const}(\vec{x})$$

$$\underbrace{(\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p}$$

w/ boundary conditions can be solved — later.

But now, let's continue and study the most general

$$\sigma_{ij} = -p\,\delta_{ij} + \underbrace{\hat{\sigma}_{ij}}$$

the thing we called "viscous" stress tensor.

(sometimes also called "deviatoric" stress tensor)

↳ or, rather, the non-isotropic part of $\hat{\sigma}$ is called that...