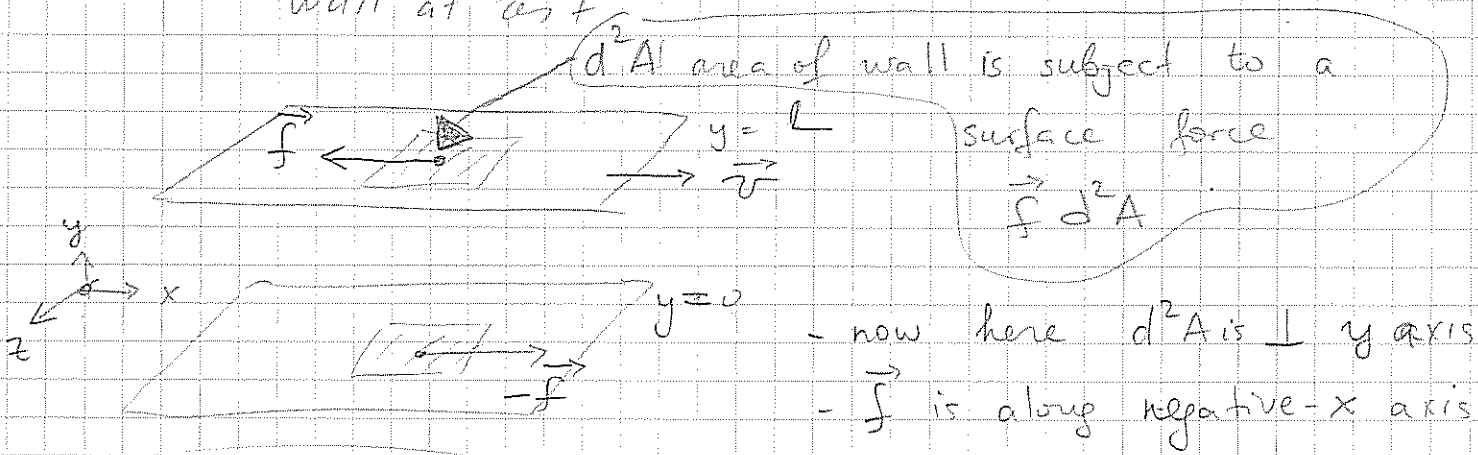
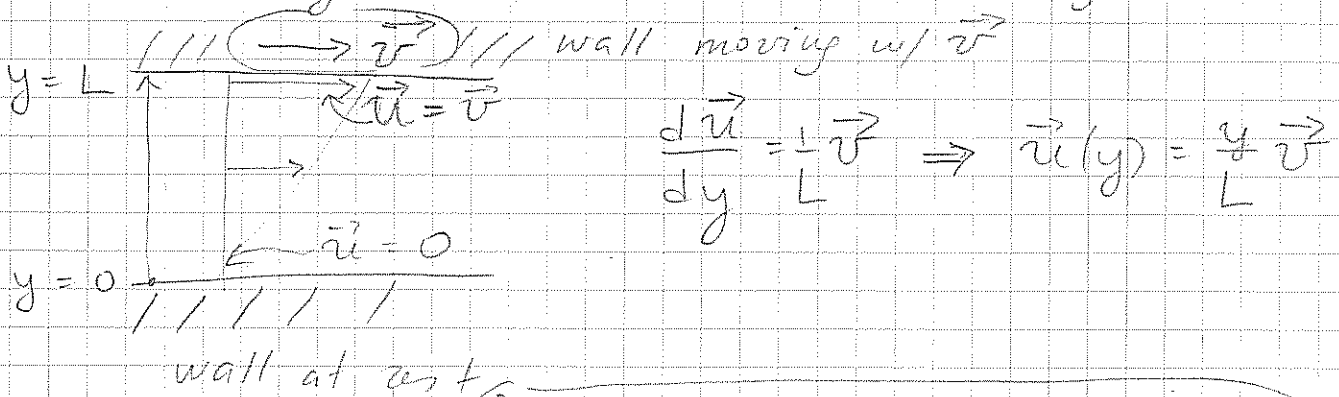


towards nonideal fluids $\Rightarrow \hat{\sigma}_{ij} = ?$

the "viscous" stress-tensor describes the surface forces that arise due to the relative motion of different parts of the fluid (like friction, basically).

A simple setting to recall what viscosity is:



d^2A area of wall is subject to a surface force $\vec{f} d^2A$

- now here d^2A is \perp y axis
 - \vec{f} is along negative- x axis

$$f_x = - \hat{\sigma}_{xy} d^2S_y$$

this function force vanishes if $\vec{v} = 0$

otherwise we have

$$\hat{\sigma}_{xy} = \mu \frac{\partial u_x}{\partial y}$$

(since f_x is force on the wall, force on fluid is w/ + sign.)

$$d^2\vec{S} = d^2A \vec{n}_A$$

$$\vec{n}_A = (0, 1, 0), \text{ i.e. along } y$$

since $\vec{u}(y) = (u_x(y), 0, 0)$, this is only nonvanishing

the coefft. μ is called the viscosity of the fluid;

$$\sigma_{xy} = \mu \frac{\partial u_x}{\partial y}$$

• is an approximation —

— there can be higher-order derivatives of $\frac{\partial^k u_x}{\partial y^k}$ on the r.h.s., but those are not so important if gradients are small enough

• it is an useful approximation, common in kinetic theory of fluid dynamic

Exs: • friction force (surface force) $\sim \hat{\sigma}_{ij} \sim$
 \sim velocity gradient

• flux of ^(solvent) particles \sim
 \sim density ^{of solvent} gradient

• flux of heat \sim
 \sim temperature gradient

also = flux of momentum: microscopic - particles moving in y direction carry x-momentum through surface \perp y

all useful 1st order approximations
 ∇ (in corresponding gradient)

the dimension of μ is

$$\hat{\sigma}_{xy} = \mu \frac{\partial u_x}{\partial y}$$

\downarrow force / area \downarrow velocity gradient = (time)⁻¹

$$\frac{N}{m^2} = [\mu] \frac{1}{s} \Rightarrow [\mu] = \frac{Ns}{m^2} = \frac{kg \cdot m \cdot s}{s^2 \cdot m^2} = \frac{kg}{s \cdot m}$$

often used $\frac{\mu}{\rho} = \frac{\text{viscosity}}{\text{mass density}} = \frac{kg \cdot m^3}{m \cdot s \cdot kg} = \frac{m^2}{s}$

does not include mass $[\frac{\mu}{\rho}] = \frac{(\text{length})^2}{\text{time}}$

also is called "kinematic viscosity"

$$\nu = \frac{\mu}{\rho}, \quad [\nu] = \frac{m^2}{s}$$

N.B. • μ, ν are useful parameterizations of friction

• microscopic origin is molecular transport in kinetic theory of gases, one shows that

$\nu \sim \sqrt{v^2} \lambda$

 $\lambda \sim \text{mean free path} \sim 10-100 \text{ nm}$

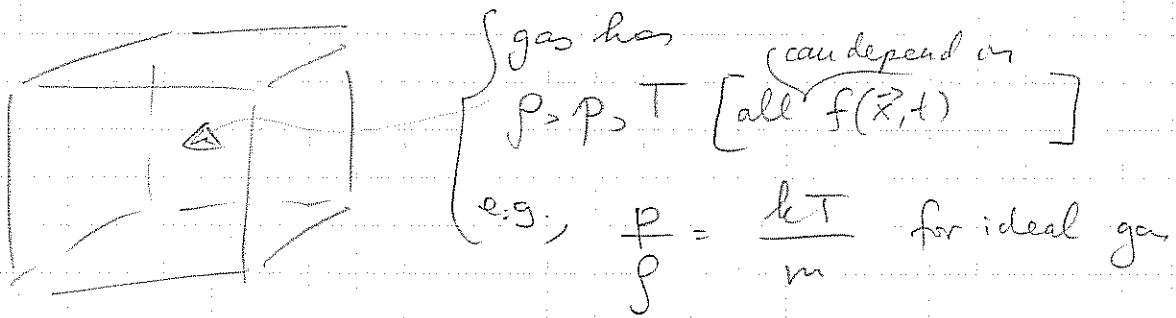
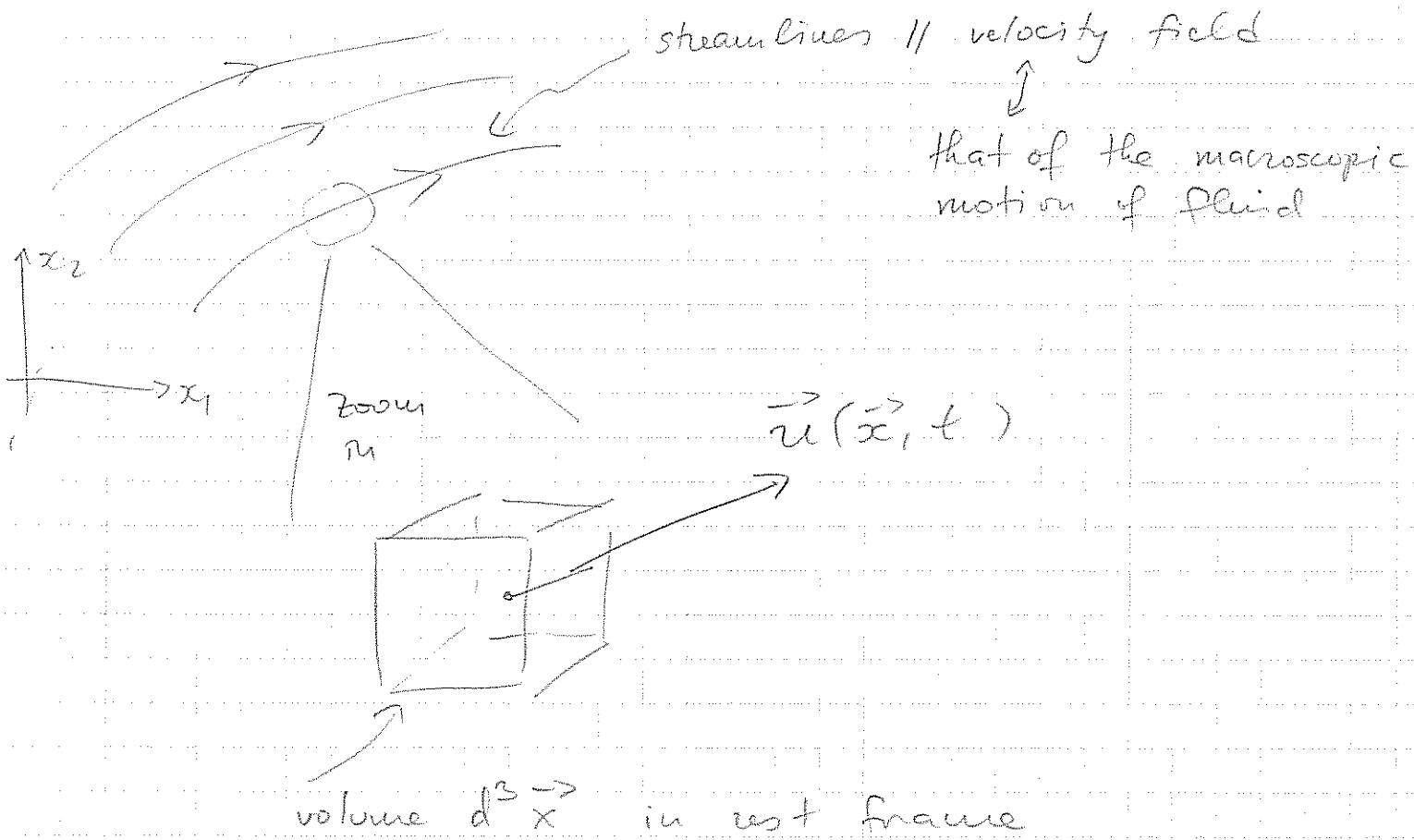
 $\sqrt{v^2} \sim \text{typical } v^2 \sim (300 \frac{m}{s})^2$

(at room air)

 $\lambda \sim \text{speed of sound}$

if you wish, by dim. analysis

recall picture in discussion of ideal fluid

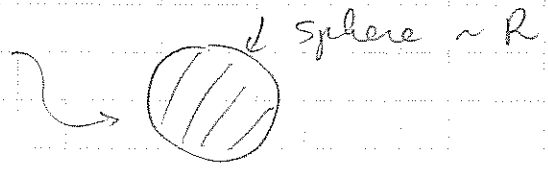


where p, T are due to the thermal motion of the particles in $d^3\vec{x}$

Now, generally, there are collisions between molecules → consider "dilute gas case"

imagine given molecule has size $\sigma \sim R^2$ ($R \sim nm$)

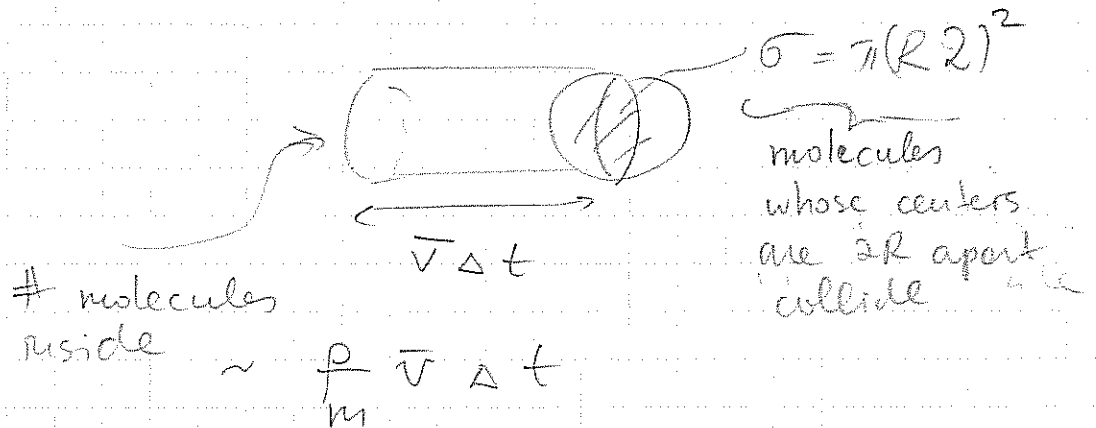
choose one at random



- let 'typical velocity' of molecules in gas be \bar{v} ($\bar{v} \sim \sqrt{kT/m} \approx \text{Maxwellian}$)
- imagine all frozen but one, moving w/ \bar{v} , it (in Δt) sweeps a cylinder of volume $(\pi R)^2 \bar{v} \Delta t$ & will collide w/ $n \sigma \bar{v} \Delta t$ molecules
 or Δt ($n = \rho/m = \# \text{ density}$) \downarrow molecule
 (this is just like calculating fluxes)

hence:

a more precise calculation uses Maxwell relative velocity distribution - "famous" $\sqrt{2}$



hence

$$\left(\begin{array}{l} \# \text{ collisions of} \\ \text{a randomly} \\ \text{chosen molecule} \\ \text{in } \Delta t \end{array} \right) \sim \frac{\rho}{m} \sigma \bar{v} \Delta t \equiv n \sigma \bar{v} \Delta t$$

($n = \# \text{ density}$)

$$\left(\begin{array}{l} \text{typical time} \\ \text{between} \\ \text{collisions} \end{array} \right) \equiv \tau \sim \frac{\Delta t}{\left(\begin{array}{l} \# \text{ collisions} \\ \text{in } \Delta t \end{array} \right)} \sim \frac{1}{n \sigma \bar{v}}$$

$$\tau \sim \frac{1}{n \sigma \bar{v}} \quad ; \text{ time between collisions}$$

$$\lambda \sim \bar{v} \tau \sim \frac{1}{n \sigma} \quad ; \text{ mean free path}$$

(distance a molecule will travel between collisions) (no kidding!)

most important formulae of kinetic theory!

Note: kinetic theory can get much more serious

than this: • replace \bar{v} \leftrightarrow actual velocity distribution = Maxwell + + nonequilibrium small corrections.

• use Boltzmann equation \nearrow (find σ)
type of potential (Lennard-Jones, say, to study approach to equilibrium

--- etc --- mostly of use to weakly nonideal gases, however...

• in the end, one can derive eqns. of FM (+ corrections)

from Boltzmann eqn ---

(recall in FM $dt \gg \tau$
~ Knudsen # layer)

we will not need any of this detail

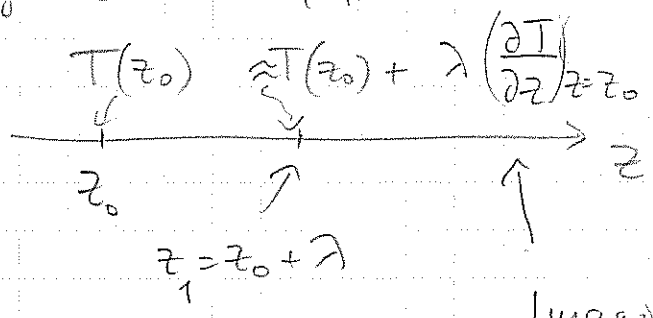
We can use elementary kinetic notions to estimate kinetic coeffs; we'll see that, for air, we'll get close to reality - justifying a posteriori our assumptions ---

- we'll assume that molecules in the gas travel on average a distance λ before they

equilibrate and start participating in the local thermal motion

- if there are gradients of ρ , n , T in the gas, molecules will transport their thermal energy ("heat") across this distance λ

E.g. imagine: $T(z_1) \approx$



Imagine $\left(\frac{\partial T}{\partial z}\right)_{z=z_0} > 0$

in average here molecules here a larger thermal

since $\bar{v}(T) \sim \sqrt{T} \sqrt{\frac{k_B}{m}}$

So, while there will be molecules near z_0 going towards z_1 , and their ^(energy) flux is

$$\sim \frac{\rho}{m} \bar{v}(z_0) \left(\frac{m \bar{v}(z_0)^2}{2}\right) \sim \rho \bar{v}(z_0)^3$$

there will also be molecules going to z_0 from z_1 , with flux

$$\sim - \frac{\rho}{m} \bar{v}(z_1) \left(\frac{\bar{v}(z_1)^2 m}{2}\right) \sim - \rho \bar{v}(z_1)^3$$

So net flux of energy is

$$j_z^E = \left(\begin{array}{l} \text{energy (heat) unit} \\ \text{crossing an area } \perp z \\ \text{in unit time} \end{array} \right) \equiv \left(\begin{array}{l} \text{energy} \\ \text{flux} \end{array} \right)$$

$$j_z^E \sim -\rho (\bar{v}(z_1)^3 - \bar{v}(z_0)^3) \quad \downarrow \text{ (we drop } 1/2, 3/2 \text{ etc here)}$$

$$\sim -\rho \bar{v}(z_0)^2 (\bar{v}(z_1) - \bar{v}(z_0))$$

$$\sim -\rho \frac{k_B T(z_0)}{m} \sqrt{\frac{k_B}{m}} (T(z_1)^{1/2} - T(z_0)^{1/2})$$

$$\sim -\rho \frac{k_B^{3/2} T(z_0)}{m^{3/2}} \frac{\lambda (\partial T / \partial z)_{z_0}}{T^{1/2}(z_0)}$$

$$j_z^E \sim -\rho \frac{k_B^{3/2}}{m^{3/2}} T^{1/2} \lambda \left(\frac{\partial T}{\partial z} \right)$$

very general story

$$\rightarrow \left(\begin{array}{l} \text{flux of} \\ \text{quantity} \\ x \end{array} \right) \equiv - \left(\begin{array}{l} \text{gradient} \\ \text{of} \\ \text{quantity } x \end{array} \right) \left(\begin{array}{l} \text{transport} \\ \text{coefft} \\ \text{for } x \end{array} \right)$$

$$\rho \rightarrow j^E = -\alpha_T \vec{\nabla} T \quad \leftarrow \text{(a factor of } k_B \text{ absorbed usually)}$$

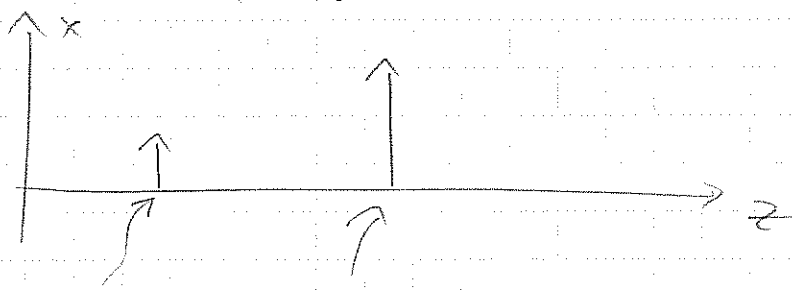
$$\text{we get } \alpha_T \sim \frac{\rho k_B^{3/2}}{m^{3/2}} T^{1/2} \lambda \sim \frac{\rho k_B^{3/2}}{m^{3/2}} T^{1/2} \frac{1}{\frac{\rho \sigma}{m}}$$

so $\lambda_T \sim k_B \sqrt{\frac{k_B T}{m}} \frac{1}{\sigma}$

$\lambda_T \sim k_B \bar{v} \frac{1}{\sigma}$

remember, we'll plug in #'s soon

For viscosity, let's set up a gradient of macroscopic velocity along z :

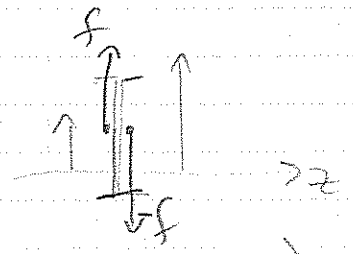


$v_x(z_0) \quad v_x(z_1) \approx v_x(z_0) + \lambda \left(\frac{\partial v_x}{\partial z} \right)_{z_0}$

$\left. \begin{matrix} j \\ j \end{matrix} \right\} \rho_i$ flux of ρ_i in j th direction
 typical velocity $\frac{m \hat{z}}{m \hat{z}} \bar{v} \sim \bar{v}$

$j_z \rho_x = \frac{\rho}{m} \left(-\bar{v} m v_x(z_0) - \bar{v} m v_x(z_1) \right)$

$= -\frac{\rho}{m} \bar{v} m \lambda \left(\frac{\partial v_x}{\partial z} \right)_{z_0}$



$j_z \rho_x = \left(\begin{matrix} \text{amount of } x\text{-momentum} \\ \text{the unit area } \perp z \\ \text{in unit time} \end{matrix} \right) \equiv \left(\begin{matrix} \text{force}_x \text{ per} \\ \text{unit area } \perp z \end{matrix} \right)$

$$\mathcal{D} \quad j_z^{p_x} = -\mu \left(\frac{\partial v_x}{\partial z} \right)$$

$$\text{so } \mu \sim \rho \bar{v}_z \lambda$$

$$\left| \nu \sim \frac{\mu}{\rho} \sim \bar{v}_z \lambda \right|$$

from p. 525

$$\left| \alpha_T \sim k_B n \bar{v} \lambda \right|$$

simple estimates
of
naive
kinetic
theory

Note: $\lambda \uparrow \rightarrow \nu \uparrow \propto \alpha_T \uparrow$

• so the more ideal a system is, the larger the kinetic coeffs are — because in an \approx ideal system transport occurs over larger distances

• v.v., the more "strongly interacting" a system is, the transport is less efficient —

we'll see #5...

Some #s to back up naive "theory" ---

air at room T

$$\bar{v} \sim 300 \frac{m}{s}$$

$$\lambda \sim 100 \mu m$$

$$\left(\rho_{air} \sim 10^{-3} \rho_{water} \approx 10^{-3} \times 1 \frac{g}{cm^3} \right)$$

$$\nu \sim \bar{v} \lambda \sim 3 \times 10^2 \frac{m}{s} \times 100 \times 10^{-9} m$$

$$\sim 3 \times 10^2 \times 100 \times 10^{-9} \frac{m^2}{s}$$

$$\sim 10^{-5} \frac{cm^2}{s}$$

$$\nu_{air \text{ room}} \sim .1 \frac{cm^2}{s}$$

"theory"

$$\nu_{air} \approx .15 \frac{cm^2}{s}$$

"expt"

→ NOT BAD for such theory.

$$\nu_{water} \sim .01 \frac{cm^2}{s} \quad (\text{--- more "strongly interacting" ---})$$

$$\nu_{oil} \sim 1 \frac{cm^2}{s}$$

$$\nu_{\text{syrup}} \sim 3 \times 10^4 \frac{cm^2}{s}$$

(don't know which)

Important numbers ---

$\frac{\mu}{k_B T}$ has dim's of h
(entropy density)

QGP // BECs
 $\frac{\mu}{k_B T} \gtrsim \frac{1}{4\pi}$
(?)

So, usually kinetic coeffs

$$\mu, \alpha_T, \left(\frac{\mu}{\rho} = \nu \right)$$

are \sim to gradients of velocity, temperature

def of "Newtonian fluid"

$$\hat{\sigma}_{ij} \sim \left\{ \frac{\partial u_k}{\partial x^l} \right\}$$

but what's the right relation?

Fluids where $\hat{\sigma} \sim \nabla u$
will be shown to obey Navier-Stokes eqs.

For non-Newtonian fluids $\hat{\sigma} \sim f(\nabla u)$,

where nonlinear terms are important

rich

- polymeric materials
- metal forming processes
- concrete, paint, chocolate
- lava flow, "solid" Earth flow
- blood flow "hemorheology"

"rheology"
studies any fluid
where Navier-Stokes
not good