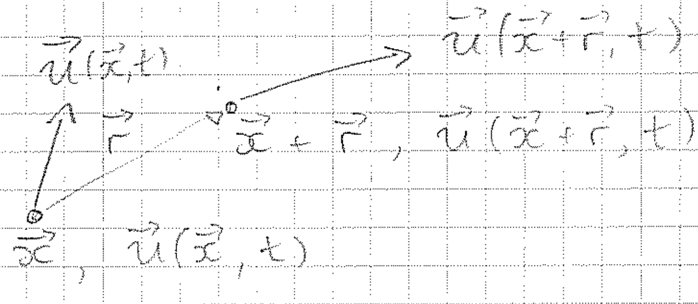
Strain, rotation, stress, & Navier-Stokes eqn

since stress ~ velocity gradients, let's 1st learn something about these.



at t — let $\vec{r}$ be near $\vec{x}$ so we can expand

$$u^1(\vec{x}+\vec{r},t) = u^1(\vec{x},t) + \sum_{i=1}^{3} \frac{\partial u^1(\vec{x})}{\partial x^i} r^i + O(\vec{r}^{\,2})$$

$$u^2 \quad = \quad 1 \to 2$$

$$u^i(\vec{x}+\vec{r},t) = u^i(\vec{x},t) + r^j \left(\frac{\partial}{\partial x^j} u^i\right)(\vec{x},t) + O(r^2)$$

derivatives of $\vec{u}$ evaluated at $\vec{x}$

in other words for small $\vec{r}$ we have

$$u^i(\vec{x}+\vec{r},t) = u^i(\vec{x},t) + \delta u^i$$

$$\delta u^i = r^j \partial_j u^i =$$

$$= r^j \tfrac{1}{2}(\partial_j u^i + \partial_i u^j) + r^j \tfrac{1}{2}(\partial_j u^i - \partial_i u^j)$$

these two cancel

In other words, I took the tensor

$\partial_j u_i$ (= 2x2 matrix)

& wrote as symmetric & antisymmetric part

$$\delta u^i = \overbrace{r^j e^{ji}}^{\delta u^i_{(S)}} + \overbrace{r^j \xi^{ji}}^{\delta u^i_{(A)}}$$

$$e^{ji} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) = e^{ij}$$

$$\xi^{ji} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\right) = -\xi^{ij} \text{ , clearly.}$$

So $\delta u^i = \delta u^i_{(S)} + \delta u^i_{(A)}$ ← again, recall this is the change of the velocity at a nearby point

Now, I can write:

$$\delta u^i_{(S)} = r^j e^{ji} =$$

$$= \frac{\partial}{\partial r^i}\underbrace{\left(\frac{1}{2} r^k r^l e^{kl}\right)}_{\frac{1}{2}\sum_{k,l=1}^{3} r^k r^l e^{kl}}$$

__Proof__: $\frac{\partial}{\partial r^i}\left(\frac{1}{2} r^k r^l e^{kl}\right) = \frac{1}{2}\delta^{ik} r^l e^{kl} + \frac{1}{2} r^k \delta^{li} e^{kl}$

$$= \frac{1}{2} r^l e^{il} + \frac{1}{2} r^k e^{ki} =$$

$$= \frac{1}{2} r^l e^{li} + \frac{1}{2} r^l e^{li} = r^l e^{li} \;\blacksquare$$

Why's that interesting?

Well, because $\delta u^i_{(s)} = \frac{\partial}{\partial r^i} \Phi$

• $\Phi = \frac{1}{2} r_k r_l e_{kl}$ is a symmetric quadratic f-n of $\{r_k\}$.

• remember your linear algebra — we've met that statement already — by a choice of an orthonormal frame, the symmetric matrix e_{kl} can always be diagonalized, which means that there exist coordinates where

$$\Phi = \frac{1}{2}\left(a\, r_1'^2 + b\, r_2'^2 + c\, r_3'^2 \right)$$

($a, b, c \equiv$ eigenvalues of $\|e_{ij}\|$)

• in this primed (orthonormal) frame, therefore,

$$\delta \vec{u}_{(s)} = (a r_1', b r_2', c r_3') \left(\equiv \frac{\partial}{\partial \vec{r}'} \Phi \right)$$

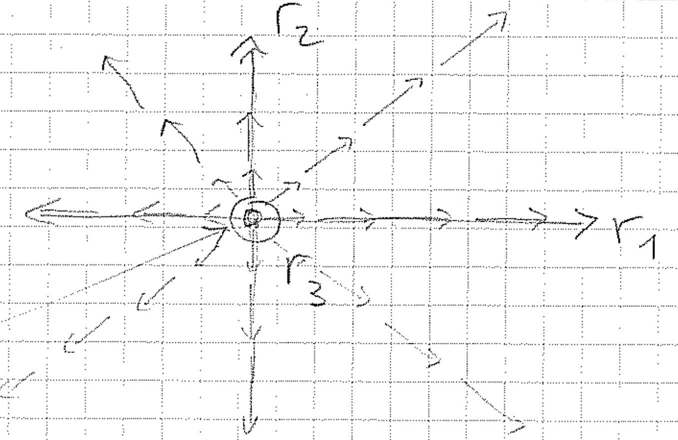
Now remember that

$\vec{u}(\vec{x}+\vec{r}) = \vec{u}(\vec{x}) + \delta\vec{u}$, let $\delta \vec{u} = \delta \vec{u}_{(s)}$ only $\longrightarrow$

then consider the velocity of fluid near $\vec{x}$ (a distance $\vec{r}$ from $\vec{x}$) relative to velocity @ $\vec{x}$ (if you wish,

go to a reference frame where $\vec{u}(\vec{x}) = 0$
(of course $\frac{\partial \vec{u}(\vec{x})}{\partial x^j}$ need not vanish, then)

So then



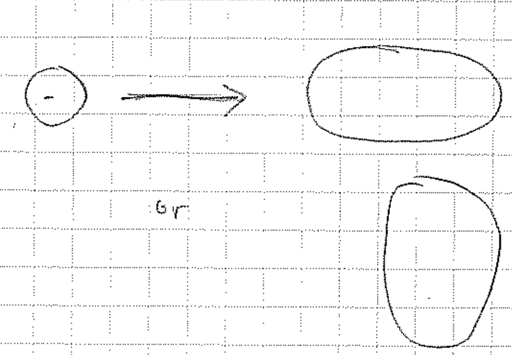
point $\vec{x}$

(can be negative)

in picture I imagined that $a = b = c$ so that "stretch" is the same in every direction

More generally, "stretch" of material near $\vec{x}$
in 1- direction occurs at a rate $a = e'_{11}$
in 2 – – – – – $b = e'_{22}$
in 3 – – – – – $c = e'_{33}$

or, equivalently a fluid element near $\vec{x}$ which is spherical is converted into ellipsoid w/ axes a, b, c



depending on orientation of principal axes of e_{ij}

e_{ij} = symmetric tensor, 6 indep elements. $\longrightarrow$

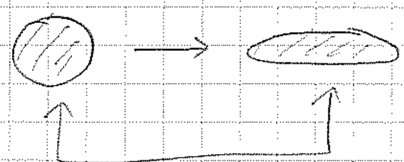
$$6 = 3 + 3$$

3 ↑ eigenvalues a, b, c

3 ↑ orientation of ellipsoid

If fluid is <u>incompressible</u> then $\vec{\nabla}\cdot\vec{v} = \partial_i v_i = e_{ii} = 0$

so $a + b + c = 0$, so we have some stretch & some compression — as we already saw



(for incompressible fluid volume doesn't change)

<u>MORAL:</u> the <u>symmetric part</u> $\delta u^i_{(S)} = r^j e^{ij}$

is determined by the

symmetric tensor $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

which is called the <u>strain tensor</u>

<u>? about the antisymmetric part ?</u>

recall $\delta\vec{u} = \delta\vec{u}_{(S)} + \delta\vec{u}_{(A)}$, $\delta u^i_{(A)} = r^j \zeta^{ji}$

$$\zeta^{ji} = \frac{1}{2}\left(\partial_j u_i - \partial_i u_j\right)$$

more linear algebra

3×3 antisymmetric matrix $= \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$

most general
3 elements.

these three elements form a vector

claim: $\forall\ \zeta_{ij} = -\zeta_{ji}$ — convention — see change from old notes !!

we have $\zeta_{ij} = \frac{1}{4}\, \varepsilon_{ijk}\, \omega_k$

or, equivalently

$$\omega_p = 2\, \varepsilon_{plm}\, \zeta_{lm}$$

Proof: = "fun w/ ε_{ijk}" (1) $\underbrace{\varepsilon_{ijk}\, \varepsilon_{lmk}}_{\Sigma \text{ over } k} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Proof: $\neq 0$ iff $i=l,\ j=m$ or $i=m,\ j=l$

sign determined by def of ϵ. ∎

(2) $\varepsilon_{imk}\, \varepsilon_{lmk} = 2\delta_{il}$.

Proof: Use (1): $\sum_m \left(\sum_k \varepsilon_{imk}\, \varepsilon_{lmk}\right) =$

$= \sum_m \delta_{il}\delta_{mm} - \delta_{im}\delta_{lm} = \delta_{il} 3 - \delta_{il} = 2\delta_{il}$. ∎

So let's , given $\zeta_{ij} = -\zeta_{ji}$,

use $\omega_p = 2\,\varepsilon_{plm}\,\zeta_{lm}$ to define $\vec{\omega}$

then $\zeta_{ij} = \frac{1}{4}\,\epsilon_{ijk}\,\omega_k$ follows from (1) :

$$\zeta_{ij} = \frac{1}{4}\times 2\,\epsilon_{ijk}\,\varepsilon_{klm}\,\zeta_{lm}$$

$$= \frac{1}{2}\left(\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\right)\zeta_{lm}$$

$$= \frac{1}{2}\left(\zeta_{ij} - \zeta_{ji}\right) = \zeta_{ij}$$

<u>More naively</u> — use $\|\zeta\| = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$

$\omega_p = 2\varepsilon_{plm}\,\zeta_{lm}$

$\Downarrow$

$\omega_1 = \cancel{\zeta_{23} - \zeta_{32}} = 4c$

$\omega_2 = \cancel{\zeta_{31} - \zeta_{13}} = -4b$

$\omega_3 = \cancel{\zeta_{12} - \zeta_{21}} = 4a$

$\boxed{\text{(end of linear algebra)}}$

back to $\delta\vec{u}_{(A)}$: $\delta u^i_A = r^j\,\zeta_j{}^i$ =

$$= \frac{1}{4}\,r^j\,\varepsilon^{jik}\,\omega^k$$

$$\delta u^i_A = \frac{1}{4} r^j \epsilon^{jik}\omega^k = -\frac{1}{4}\epsilon^{ijk} r^j \omega^k = +\frac{1}{4}\epsilon^{ikj}\omega^k r^j = \frac{1}{4}(\vec{\omega}\times\vec{r})^i$$

in vector not'n: $$\delta \vec{u}_{(A)} = \frac{1}{4}\,\vec{\omega}\times\vec{r}$$

since $\omega_p = 2\epsilon_{plm}\xi_{lm} = 2\cdot\frac{1}{2}\epsilon_{plm}(\partial_l u_m - \partial_m u_l)$

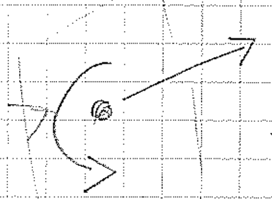
$$\left[\;\omega_p = \epsilon_{plm}(\partial_l u_m - \partial_m u_l) \qquad \vec{\omega} = \vec{\nabla}\times\vec{u}\;\right]$$

but this means that $\delta\vec{u}_{(A)}$ corresponds to a rigid body rotation w/ angular velocity $\vec{\omega}\,\frac{1}{4}$ determined by $\vec{\nabla}\times\vec{u} = \vec{\omega}$

$\vec{\omega} = \vec{\nabla}\times\vec{u}$ is called the <u>vorticity</u> of the fluid at $\vec{x}$ (recall derivative evaluated @ $\vec{x}$)

if $\delta\vec{u} = \delta\vec{u}_{(A)}$ we have,

relative to the point $\vec{x}$



$\vec{\omega} = \vec{\nabla}\times\vec{u}$

= rotation around an axis $\vec{\omega}$ w/ angular velocity $|\vec{\omega}|\frac{1}{4}$

$\vec{x}$

putting it all together

$$u^i(\vec{x}+\vec{r}) = u^i(\vec{x}) + \frac{\partial}{\partial r^i}\left(\frac{1}{2} r_k r_l e_{kl}\right)$$

$$+ \frac{1}{4}\epsilon_{ijk}\,\omega_j\, r_k$$

(motion of fluid @ $\vec{x}+\vec{r}$) =

= (uniform motion w/ $u^i(\vec{x})$) +

+ (straining motion w/ rate-of-strain tensor $e_{ij} = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}$)

+ (rigid-body rotation w/ $\frac{1}{4}\vec{\omega}$, $\vec{\omega} = \nabla \times \vec{u}$ vorticity)

(straining motion itself) = (isotropic strain (trace part of e_{ij})) + (anisotropic strain w/out change of volume (traceless part of e_{ij}))

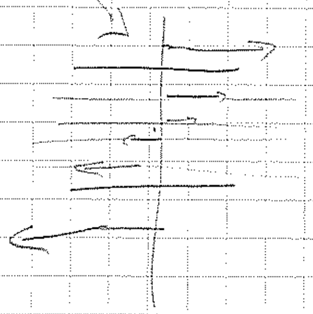
(this is NOT there for incompressible fluid as $e_{ii} = 0$)

Remarks

① "simple shearing motion"

- $\delta\vec{u}$ has same direction everywhere
- $|\delta\vec{u}|$ varies in a direction $\perp$ to $\delta\vec{u}$

HW 2. Pr. 1 : ∘ find principal rates of strain (a, b, c)
& the principal axes of strain w.r.t. $\delta\vec{u}$

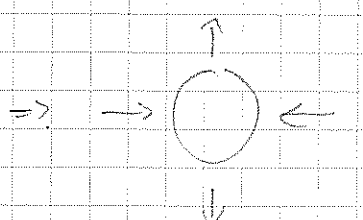


∘ show simple shearing motion ≡ strain + rotn (explain how)

∘ show: any local relative velocity field =
= symm. expansion + two simple shearing motns + rigid rotn.

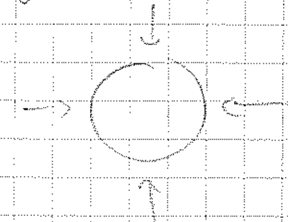
② incompressible fluid $\vec{\nabla}\cdot\vec{u} = \partial_i u_i = 0 \Rightarrow e_{ii} = 0 \Rightarrow$
$\Rightarrow a + b + c = 0$

(also made on p. 58 sorry)



so volume preserved (2d picture)

NEVER:



which changes volume

③ vorticity will play a role in what follows —
— when we start analyzing problems.

Finally, Navier-Stokes eqns

we said $\sigma_{ij} = -p\delta_{ij} + \hat{\sigma}_{ij}$

& $\hat{\sigma}_{ij} \sim \partial_k v_\ell \longrightarrow$ how?

we said that $\partial_k v_\ell = e_{k\ell} + \zeta_{k\ell}$

(p. 54-55)

$$= \left(e_{k\ell} - \frac{1}{3}\delta_{k\ell}\, e_{mm}\right) + \frac{1}{3}\delta_{k\ell}\, e_{mm} + \frac{1}{2}\epsilon_{k\ell m}\,\omega_m$$

$\partial_k v_\ell$ - (9 elements)

$e_{k\ell} - \frac{1}{3}\delta_{k\ell}\, e_{mm}$ (5 elements) $\longrightarrow$ three independent tensor structures:
- symmetric traceless (strain w/out isotropi

$\frac{1}{3}\delta_{k\ell}\, e_{mm}$ (1 element) $\longrightarrow$ - isotropic (expansion/compression)

$\frac{1}{2}\epsilon_{k\ell m}\,\omega_m$ (3 element) $\longrightarrow$ - antisymmetric (rotation)

(all three tensors are $\sim \partial v$)

the importance of these three structures is that, for a fluid that itself does not have any directional structure (no long, polar etc molecules) — e.g an isotropic fluid (at rest, looks the same in every direction) — it must be that:

$$\hat{\sigma}_{ij} = A\left(e_{ij} - \delta_{ij}\frac{e_{\ell\ell}}{3}\right) + B\, e_{\ell\ell}\,\delta_{ij} + C\epsilon_{ij\ell}\,\frac{1}{2}\omega_\ell$$

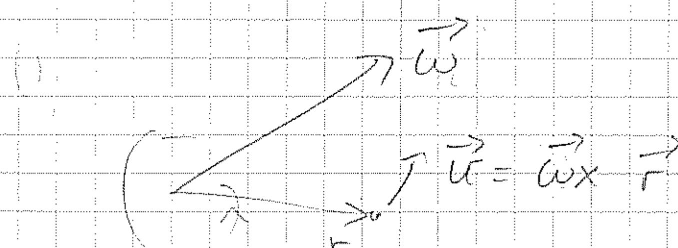
i.e. be "covariant" under rotations

In other words, the relation between $\hat{\sigma}_{ij}$ and the strain & $\vec{\omega}$ should look the same no matter how the coordinate system is oriented (remark on this: three & only three tensor structures) !

Now, consider a fluid in motion such that $e_{ij} = 0$ but $\omega_\ell \neq 0$

& constant

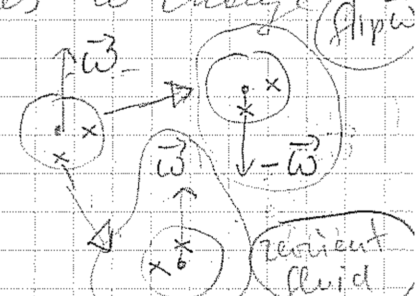
i.e. take $\vec{u} = \underset{\text{constant}}{\vec{\omega}} \times \vec{r}$, a fluid in uniform rotation around some axis:



this is yet another (HW#2) proof of symmetry of σ_{ij}

Suppose we flip sign of $\vec{\omega}$ ⟶ leads to change of sign of $\hat{\sigma}_{ij} \approx C \epsilon_{ijk} \omega_k$.

$$\hat{\sigma} \longrightarrow -\hat{\sigma}$$



But for an isotropic fluid, this is equivalent to just changing orientation (w/ $\vec{\omega}$-fixed) and $\hat{\sigma}$ shouldn't change upon doing this. (Stokes argued that since pure rotation doesn't deform a fluid element, there should be no friction associated with it, so $C = 0$).

So, we have, renaming $A \to 2\mu$, $B \to \lambda$ → 2nd viscosity

$$\hat{\sigma}_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}\, e_{kk}\right) + \lambda\, \delta_{ij}\, e_{kk}$$

both > 0 (HW 2)

our already known viscosity

(easy to see, let only $\frac{\partial v^y}{\partial x^x} \neq 0$)

$$\frac{\partial}{\partial t}(\rho u_i) = \rho F^i - \partial_j\left(\rho u^i u^j + p\,\delta^{ij} - \hat{\sigma}_{ij}\right)$$

$$= \rho F_i - \partial_j\Big(\rho u^i u^j + p\,\delta^{ij} - 2\mu\left(e^{ij} - \frac{1}{3}\delta^{ij} e^{kk}\right) - \lambda\, \delta_{ij}\, e^{kk}\Big)$$

$$\boxed{(\rho u_i)^{\cdot} = \rho F_i - \partial_j\left[\rho u^i u^j + \left(p + \left(\frac{2}{3}\mu - \lambda\right) e^{\ell\ell}\right)\delta_{ij} - 2\mu\, e_{ij}\right]}$$

μ & λ – the two viscosity coeff's can depend on ρ, T
– generally not uniform, so can't assume $\partial_j(\mu \text{ or } \lambda) = 0$

If μ & λ are $\approx$ constant – $\vec{x}$ independent, can take out of ∂_j, also $e^{\ell\ell} = \vec{\nabla}\cdot\vec{u}$

$$\text{\&} \quad \partial_j e_{ij} = \partial_j \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) = \frac{1}{2}\left(\vec{\nabla}^2 u^i + \partial_i(\vec{\nabla}\cdot\vec{u})\right)$$

recall that $(\rho u_i)^\circ = \dot{\rho} u_i + \rho \dot{u}_i = -\partial_j(\rho u^j) u^i + \rho \dot{u}_i$

so then from p 66 we have

$$\rho \dot{u}_i - \underline{u_i \partial_j(\rho u^j)} = \rho F_i - \underline{\partial_j(\rho u^j) u^i} - \rho u^j \partial_j u^i$$
$$- \partial_i p - \left(\tfrac{2}{3}\mu - \lambda\right)\partial_i e_{ll} + 2\mu \partial_j e_{ij}$$

$$\rho\left(\frac{\partial}{\partial t} u_i + (\vec{u}\cdot\nabla) u_i\right) = \rho F_i - \partial_i p - \left(\frac{2\mu}{3} - \lambda\right)\partial_i(\vec{\nabla}\cdot\vec{u})$$
$$+ \mu \vec{\nabla}^2 u_i + \mu \partial_i(\vec{\nabla}\cdot\vec{u})$$

$$\boxed{\rho(\dot{u}_i + \vec{u}\cdot\vec{\nabla} u_i) = \rho F_i - \partial_i p + \mu \vec{\nabla}^2 u_i + \left(\lambda + \frac{\mu}{3}\right)\partial_i(\vec{\nabla}\cdot\vec{u})}$$

or in vector notn:

$$\boxed{\rho(\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u}) = \rho\vec{F} - \vec{\nabla}p + \mu \Delta\vec{u} + \left(\lambda + \frac{\mu}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})}$$

▷ Navier-Stokes eqn
* 1827 Navier (conjectured it, basically)
* 1845 Stokes derived
  ≈ like we do, but no λ

If fluid is incompressible - only one viscosity, μ:

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p + \frac{\mu}{\rho}\Delta\vec{u}$$

$$\boxed{\dot{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p + \nu\Delta\vec{u}}$$

Navier-Stokes, incompressible fluid