Now back to our energy balance of fluid

recall energy density per unit mass =

$$= \frac{\vec{u}^2}{2} + \varepsilon$$

kinetic — internal energy per unit mass

this is the one obeying

$$d\varepsilon = T\,ds - p\,dV$$

or $$d\varepsilon = T\,ds + \frac{p}{\rho^2}\,d\rho$$ see pp 43 - 44

where "d" refers to
the change appropriate to a physical fluid element, i.e. when considering time variation, this becomes

$$\frac{D\varepsilon}{Dt} = T\,\frac{Ds}{Dt} + \frac{p}{\rho^2}\,\frac{D\rho}{Dt}.$$

Now, on a fluid element inside some fixed V
work of body forces is done at a rate

$$\int_V u_i F_i \rho\, dV$$

work of surface ones is at a rate:

$$\int_{\partial V = S} u_i \sigma_{ij}\, d^2 S_j = \int_V \partial_j (u_i \sigma_{ij})\, dV$$

∴ rate of work done on fluid in V is

$$\int_V dV \left(\rho u_i F_i + u_i \frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij} \frac{\partial u_i}{\partial x_j} \right)$$

again, this obeys Thermo Dynamic relations so we divide by ρ

or, dropping $\int_V dV$,

rate of work on fluid element in d^3x, per unit ma

$$\underbrace{u_i F_i + \frac{u_i}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j}} + \frac{1}{\rho} \sigma_{ij} \frac{\partial u_i}{\partial x_j}$$

$$u_i \left(F_i + \frac{u_i}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} \right)$$

but recall $\rho \frac{D u_i}{D t} = \rho F^i + \frac{\partial \sigma_{ij}}{\partial x_j}$

our "$\bar{f} = m\bar{a}$" equ of way back!!

p. 15, equiv. to top of 26

(equiv. to Navier-Stokes if σ_{ij} given)

so rate of work on fluid element @ $\vec{x}$ per unit mass is $u_i \left(F_i + \frac{u_i}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j} \right) + \frac{1}{\rho} \sigma_{ij} \frac{\partial u_i}{\partial x_j} =$

$$= u_i \frac{Du_i}{Dt} + \frac{1}{\rho}\, \sigma_{ij} \frac{\partial u_i}{\partial x_j}$$

This is change of kinetic energy per unit mass of a physical volume element

This is change of internal energy per unit mass — if $\sigma_{ij} = -\delta_{ij} p$ this is $\left(-\frac{p}{\rho} \vec{\nabla} \vec{u}\right)$ — as we already know from discussion of ideal fluid energy balance, p 46 of notes

So $\frac{1}{\rho} \sigma_{ij} \frac{\partial u_i}{\partial x_j}$ generalizes the rate of change of internal energy per unit mass ε due to compression (to which it reduces to for ideal fluid) to that due to compression + viscosity (inside $\hat{\sigma}_{ij}$)

but internal energy can also change because of heat (in/out) flow:

$$\underbrace{\int_{\partial V = S} \underbrace{\varkappa_T \frac{\partial T}{\partial x_i}}_{i\text{-th component of heat flux}} d^2 S^i}_{\text{heat flux through } S = \partial V} = \int_V dV \underbrace{\frac{\partial}{\partial x^i}\left(\varkappa_T \frac{\partial T}{\partial x^i}\right)}$$

rate of increase of internal energy of fluid @ $\vec{x}$: i.e. increase of $(\varepsilon\rho)$ due to heat flux — increase of ε is $\frac{1}{\rho}$ that amount

So we add the two sources of change of ε, internal energy per unit mass:

$$\frac{D\varepsilon}{Dt} = \frac{1}{\rho}\sigma_{ij}\frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\varkappa_T \frac{\partial T}{\partial x^i}\right)$$

now, $\sigma_{ij} = -p\,\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}e_{\ell\ell}\right) + \lambda\,\delta_{ij}\,e_{\ell\ell}$

leads to

$$\frac{D\varepsilon}{Dt} = -\frac{p}{\rho}\underbrace{\vec{\nabla}\cdot\vec{u}}_{e_{\ell\ell}} + \frac{2\mu}{\rho}\overbrace{e_{ij}\frac{\partial u_i}{\partial x_j}}^{=\,e_{ij}e_{ij}} - \frac{2}{3}\frac{\mu}{\rho}\overbrace{\vec{\nabla}\cdot\vec{u}}^{=\,e_{\ell\ell}}e_{\ell\ell}$$

$$+ \frac{\lambda}{\rho}\underbrace{\vec{\nabla}\cdot\vec{u}}_{e_{\ell\ell}}e_{\ell\ell} + \frac{1}{\rho}\partial_i\left(\varkappa_T\,\partial_i T\right)$$

$$\frac{D\varepsilon}{Dt} = \underbrace{-\frac{p}{\rho}e_{\ell\ell}}_{\text{deformation work}} + \underbrace{\frac{2\mu}{\rho}e_{ij}e_{ij} + \left(\lambda - \frac{2}{3}\mu\right)\frac{1}{\rho}\left(e_{\ell\ell}\right)^2}_{\text{dissipation due to viscosity}}$$

$$+ \underbrace{\frac{1}{\rho}\partial_i\left(\varkappa_T\,\partial_i T\right)}_{\text{heat flux dissipation}}$$

HW2, Pr. #2:

$$\rho T\frac{Ds}{Dt} = \vec{\nabla}\left(\varkappa_T\vec{\nabla}T\right) + 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}e_{\ell\ell}\right)^2 + \lambda\left(e_{\ell\ell}\right)^2$$

+ argue „ $Ds \geq 0 \Rightarrow \varkappa_T \geq 0,\ \mu > 0,\ \lambda > 0$

Now we can combine all & say that:

(1) $\mathring{\rho} + \vec{\nabla}(\vec{u}\rho) = 0$ ⟸ correct!! ordinary!! TWO

(2)-(4) $$\rho\left(\mathring{\vec{u}} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right) = \rho\vec{F} - \vec{\nabla}p + \mu\Delta\vec{u} + \left(\lambda + \frac{\mu}{3}\right)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})$$

(5) $$\rho T \frac{Ds}{Dt} = \varkappa_T \Delta T + 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}e_{\ell\ell}\right)^2 + \lambda(e_{\ell\ell})^2$$

(6) $f(p, \rho, T) = 0$ – equ. of state, include $s = s(p, T)$, say.

$\underbrace{\vec{u}, p, \rho, T}$ → van der Waals – OVER

6 variables $\bar{x}, t$ dependent +

& 6 equations

complete set of FD equations

$\varkappa_T, \mu, \lambda$ = 3 kinetic coefficients (above, assumed $\bar{x}$-t independent; but didn't need to)

on Thermodynamic quantities:

- mass density ρ – unambiguous
- internal energy per unit mass obeys $d\mathcal{E} = \delta Q + \delta W$, 1st law (energy in comoving frame of fluid)
- $\underline{T}$ & $\underline{s}$ & $\underline{p}$ defined now using thermodynamic relation appropriate to the equilibrium system w/ given ρ, $\mathcal{E}$
- while $\underline{s}$ is not the "real" thermodynamic entropy, to the linear order in $\partial_i V_j$ we're working, it does coincide w/ it ((deviations are quadratic in ∇v, so $Ds > 0$. HW2#2))

- the "p" entering Navier-Stokes equ. is the one appropriate to equilibrium. In a moving fluid, the tangential stress (what we normally perceive as pressure) has an extra term, coming from part of $\sigma_{ij} \sim \delta_{ij}$

recall
$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\left(e_{ij} - \tfrac{1}{3}\delta_{ij}\, e_{\ell\ell}\right) + \lambda\, e_{\ell\ell}\,\delta_{ij}$$

the isotropic part of σ_{ij} is

$$\sigma_{ij}\Big|_{\text{isotropic}} = -\underbrace{(p - \lambda e_{\ell\ell})}\,\delta_{ij}$$
(HW 2, problem 2

so $p' = p - \lambda\, \vec{\nabla}\cdot\vec{v}$, $\lambda > 0$

"real" pressure, i.e. isotropic part of stress

"bulk" viscosity upon compression/expansion ($\nabla v \neq 0$)

leads to sound waves attenuation ('cause dissipative)

real pressure $\neq$ equilibrium pressure, (to linear order in ∇v, as opposed to S — for ex.)

* expansion $p < p_{\text{equil}}$ } sensible
* compression $p > p_{\text{equil}}$

in most "normal" FM problems λ will not be important – mostly for reasons that most ("normal") fluids are usually considered to be incompressible

$\Downarrow$

* formally, this simplifies equations, as $\bar{\nabla}\cdot\bar{v} = 0$

* physically, the following conditions need to be obeyed for "$\bar{\nabla}\cdot\bar{v} = 0$" to be a good approximation:

(a.) $v \ll c$

typical velocity of fluid motion

speed of sound in fluid, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

we'll come to that

constant s (entropy per unit mass)

(b.) $\tau \gg \frac{L}{c}$

for non-stationary flows only, of course

typical length dimension associated w/ fluid flow (i.e. typical distance scale where flow changes appreciably $O(1)$)

typical time scale on which fluid flow changes appreciably

ratio $\frac{v}{c}$ is often called "Mach number"

(c.) some extra conditions involving μ & $\varkappa_T$ → usually ok, later
(for viscous fluids)

So, e.g. in air where $c_{air} \sim 340 \frac{m}{s}$

($c_{water} \sim 1500 \frac{m}{s}$ at 0 – 80°C) ($\sim 1100 \frac{km}{h}$ or so)

it appears reasonable that $\bar{\nabla} \cdot \bar{v} = 0$ will be OK.

Now, the way (a) is arrived at is by estimating the change of density $\Delta \rho$ of a fluid in a stationary motion w/ $\vec{u}$ and demanding that $\frac{\Delta \rho}{\rho} \ll 1$ → which for a physicist is a good enough reason to say $\Delta \rho \approx 0$. To do this, the easiest way is through Bernoulli's law, which we'll learn about next (as well as c).

But first, our plans:

* we've got the most general eqns. describing Newtonian fluids – including $\mu, \lambda, \varkappa_T$ effects.

* $\mu, \lambda, \varkappa_T \neq 0$ is too general & too difficult!
  - this is why people often start FM courses by studying ideal fluids – many more problems can be solved for their flows
  - however! – solutions so obtained are not always a fair representation of physics – since viscosity μ is important, near boundaries, for example.

(it's all many math FM courses do)

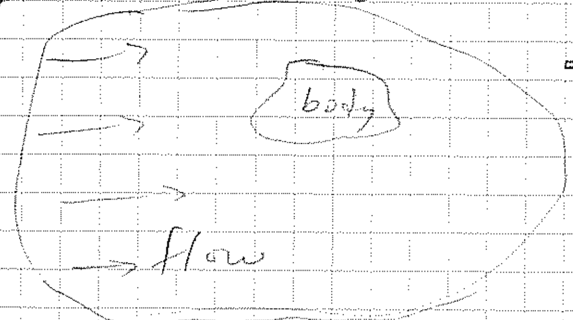
(very!)

* so our strategy will be to study some viscous incompressible fluid flows & understand when/where viscosity can be ignored (and where it can't)

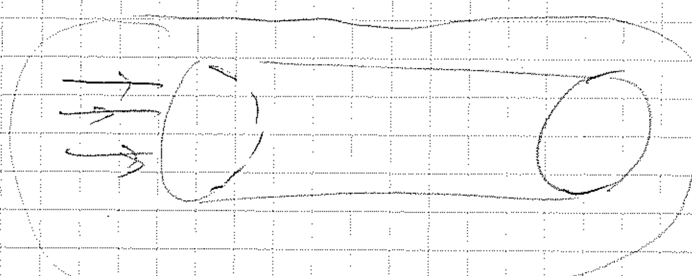
* then, we'll do some ideal fluid flows and also get an idea about stability of flows and other related problems

* if we've got time left, we'll also study some turbulence, too --

Some problems we'd like to get an appreciation of are, e.g.:



- recall HW 1 Problem 1 —
- what is the force on body? - drag? - lift? (airplanes)



- flow thru a pipe -- -- stability?

- sound waves --
- gravity waves --

Bernoulli's law:

- *idea*, recall Newton's law $m\ddot{x} = f$ & let $f = -\frac{\partial}{\partial x} V(x)$ time-independent potential

$$\rightarrow \quad m\ddot{x} + \frac{\partial}{\partial x} V(x) = 0 \quad \Big| \; x \quad \dot{x}$$

$$m\dot{x}\ddot{x} + \frac{\partial V}{\partial x}\dot{x} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2\right) + \frac{d}{dt} V(x) = 0$$

so $$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + V(x)\right) = 0$$

& $\frac{m\dot{x}^2}{2} + V(x) = \text{const} \iff$ integral of motion (energy), constant on particle trajectories

- "Bernoulli's law" = similar story for fluid motion.
  - energy per unit mass of a fluid element is: $\frac{1}{2}\vec{u}^2 + \varepsilon$
  - $\frac{D}{Dt}\left(\frac{\vec{u}^2}{2} + \varepsilon\right)$ = rate of work of body & surface forces per unit mass + rate of heat flow per unit mass

recall deformation work cancels between kinetic & internal

as per eqns. on pp 74-77 $\longrightarrow$ $$= u_i F_i + \frac{1}{\rho}\frac{\partial(u_i \sigma_{ij})}{\partial x^j} + \frac{1}{\rho}\frac{\partial}{\partial x^i}\left(\varkappa_T \frac{\partial T}{\partial x^i}\right)$$

Next, we'll assume that F_i is a potential force per unit mass, i.e.

$$F_i = -\partial_i \Psi \quad , \quad \text{i.e.} \quad F_i = -\frac{\partial}{\partial x^i}(gz) = -g\delta_i$$

$(z = x^3)$

for semblance w/ Newton's case

and that Ψ is t-independent, only $\vec{x}$-dependent

then $u_i F_i = -u_i \frac{\partial}{\partial x_i}\Psi = -\vec{u}\vec{\nabla}\Psi =$

$$= -\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\Psi = -\frac{D\Psi}{Dt} .$$

the term we write as: $\frac{1}{\rho}\frac{\partial}{\partial x_j}(u_i \sigma_{ij}) = -\frac{1}{\rho}\frac{\partial}{\partial x_j}(u_j p) + \frac{1}{\rho}\frac{\partial}{\partial x_j}(u_i \hat{\sigma}_{ij})$

$$= -\frac{1}{\rho}\underbrace{\frac{\partial u_j}{\partial x_j}}\, p - \frac{1}{\rho} u_j \frac{\partial p}{\partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x_j}(u_i \hat{\sigma}_{ij}) = (*)$$

use $\frac{\partial \rho}{\partial t} = -\vec{\nabla}(\rho\vec{u}) = -\vec{u}\vec{\nabla}\rho - \rho\vec{\nabla}\vec{u} \Rightarrow$

$\Rightarrow \vec{\nabla}\vec{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$

$$(*) = \underbrace{+\frac{1}{\rho^2}\frac{D\rho}{Dt}p \underbrace{- \frac{1}{\rho}\vec{u}\vec{\nabla}p - \frac{1}{\rho}\frac{\partial p}{\partial t}}_{-\frac{1}{\rho}\frac{Dp}{Dt}}} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\partial_j(u_i\hat{\sigma}_{ij})$$

$$= -\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\partial_j(u_i\hat{\sigma}_{ij})$$

back to equ. on bottom of 82; results for $\vec{u}\cdot\vec{F}$ & 2nd term

$$\frac{D}{Dt}\left(\frac{\vec{u}^2}{2}+\varepsilon\right) = -\frac{D\psi}{Dt} - \frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\left(\frac{\partial p}{\partial t} + \partial_j\left(u_i\hat{\sigma}_{ij} + \varkappa_T\partial_j\right.\right.$$

$$\frac{D}{Dt}\left(\frac{\vec{u}^2}{2}+\varepsilon+\psi+\frac{p}{\rho}\right) = \frac{1}{\rho}\left(\frac{\partial p}{\partial t} + \partial_j\left(u_i\hat{\sigma}_{ij} + \varkappa_T\,\partial_j T\right)\right)$$

(this expresses "Bernoulli's law" in more generality than he derived it)

What it says is that if r.h.s. $= 0 \Rightarrow$

$$\frac{D}{Dt}\underbrace{\left(\frac{\vec{u}^2}{2}+\varepsilon+\psi+\frac{p}{\rho}\right)} = 0$$

so this quantity is <u>constant</u> for a <u>physical fluid element</u>

$\left(\frac{D}{Dt}!\right)$

$\rightarrow$ (akin to energy of a Newtonian particle (= const along the trajectory

<u>Qu</u>:

<u>When does Bernoulli's law hold?</u>

<u>Answer</u>: (1) when rhs $= 0 \Rightarrow \underbrace{\varkappa_T, \mu, \lambda = 0}_{\text{ideal fluid}}$

(2) $\frac{\partial p}{\partial t} = 0 \Rightarrow p = p(\vec{x}, \cancel{t}) \longleftarrow$ no t-dep.

<u>pressure field is stationary</u>

N.B.: no assumption was made that fluid is incompressible (Bernoulli assumed this) 1783!

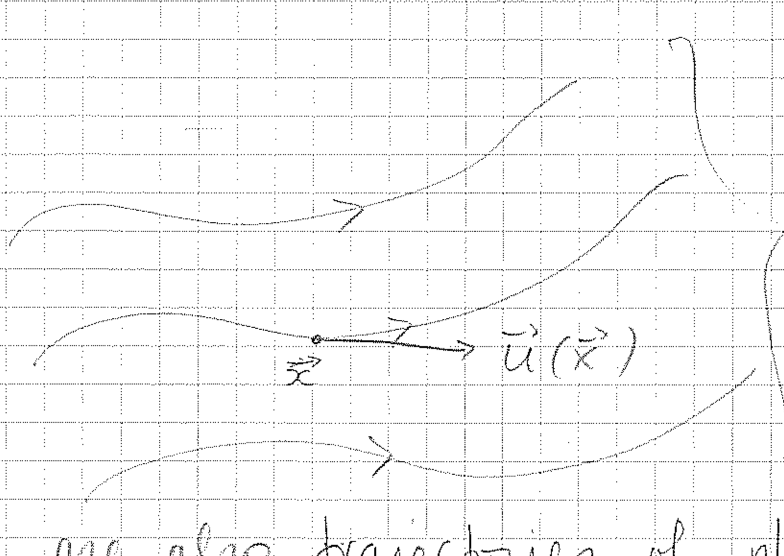
Ψ & $\frac{p}{\rho}$ play a role of potential energies per unit mass

potential of body force

potential of surface force due to isotropic pressure.

both time-independent!

Usually, if pressure field is steady, also velocity field is steady; i.e. $\underbrace{\vec{u}(\vec{x}, \cancel{t})}_{\text{does not contain explicit } t \text{ dependence}}$



for a steady flow streamlines (lines whose tangent $\forall t$ is $\equiv \vec{u}(\vec{x})$) are also trajectories of physical fluid elements

(put some dye in a fluid @ steady flow - die moves along streamlines

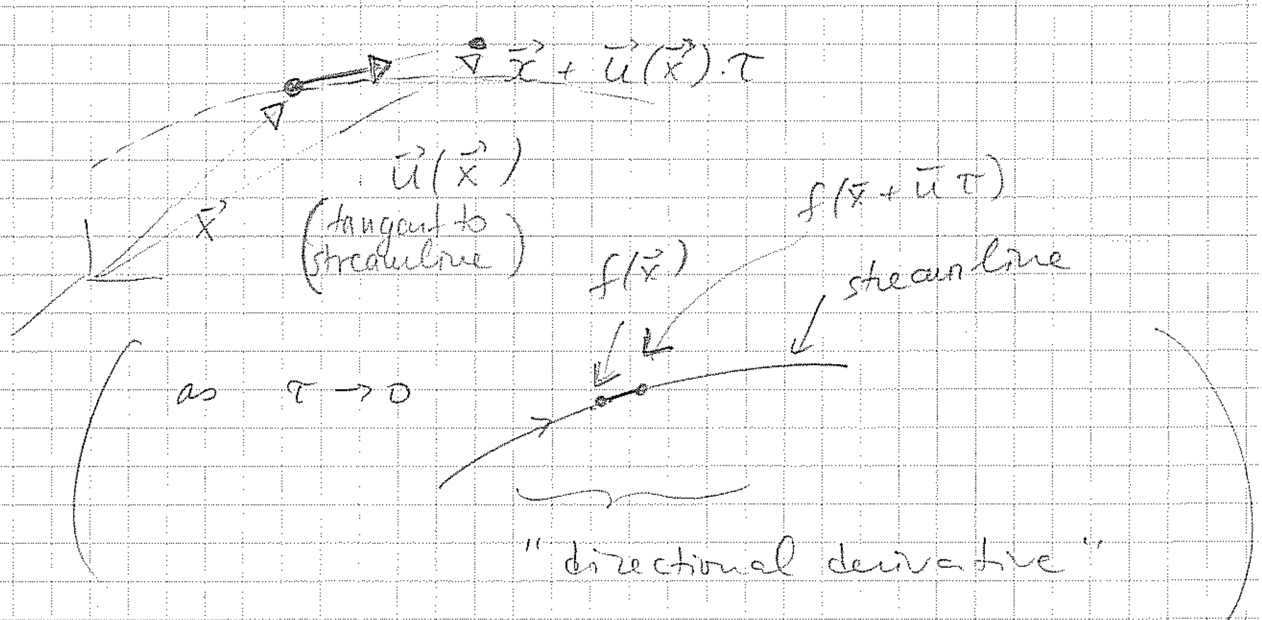
For a steady flow $\frac{D}{Dt} = (\vec{u}\cdot\vec{\nabla})$

Bernoulli then says:

$$\underbrace{\vec{u}\cdot\vec{\nabla}}_{\text{derivative along streamline}}\left[\frac{1}{2}\vec{u}^2 + \varepsilon + \psi + \frac{p}{\rho}\right] = 0$$

– derivative along streamline

e.g. $$\vec{u}\cdot\vec{\nabla} f(\vec{x}) = \left.\frac{f(\vec{x} + \vec{u}(\vec{x})\tau) - f(\vec{x})}{\tau}\right|_{\tau\to 0}$$



So, then we express <u>Bernoulli's law</u> as:

$$\frac{1}{2}\vec{u}^2 + \varepsilon + \psi + \frac{p}{\rho} = \text{const.}$$

<u>along streamlines, for an ideal fluid flow,</u>
<u>which is steady subject to potential force.</u>

[Tolley said: "forget about magnetic forces — we do not consider conductive fluids here"

If the fluid is incompressible, can drop $\mathcal{E}$, as internal energy is then separately conserved (no exchange between kinetic & internal due to deformation work); so for an incompressible fluid Bernoulli's law is usually stated by saying "$\frac{\vec{u}^2}{2} + \frac{p}{\rho} + \Psi = \text{const}$ along the streamlines (= particle trajectories) for an incompressible, inviscid, steady fluid flow, subjected to a body force per unit mass $F_i = -\partial_i \Psi$"

(Bernoulli will be of use also for viscous flows in situations where viscosity can be $\simeq$ ignored).

Now recall our discussion that we can take ρ & s as parameters of state. Then $p = p(\rho, s)$ gives the eqn of state & so

$$\Delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \Delta \rho + \left(\frac{\partial p}{\partial s}\right)_\rho \Delta s$$

If flow is isentropic (like it would be for an ideal fluid, see pp 41-74), this implies that for a physical fluid