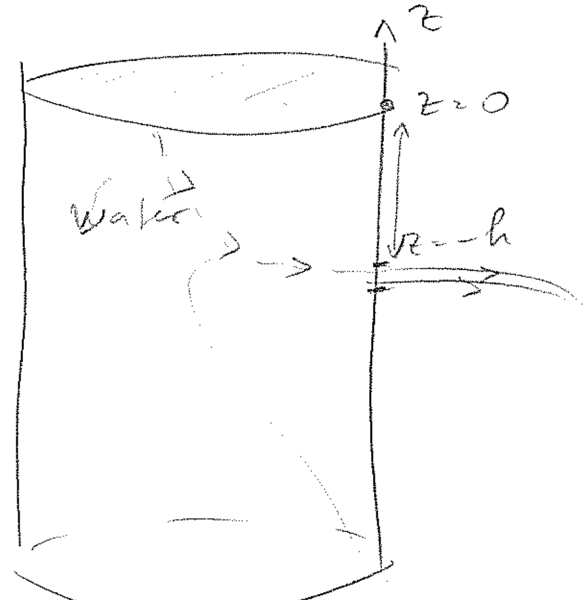
Application 1



Torricelli

imagine streamline .. $\vec{u} \approx 0$ @ surface

$|\vec{u}| = ?$ @ hole

$$\left(\frac{\vec{u}^2}{2} + \cancel{\varepsilon} + \Psi + \frac{p}{\rho}\right)_{surface} = \left(\frac{u^2}{2} + \cancel{\varepsilon} + \Psi + \frac{p}{\rho}\right)_{near\ hole}$$

incompressible ideal

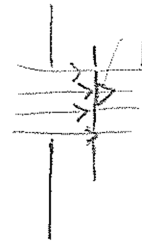
$$\Psi: \quad -\vec{\nabla}\Psi = \vec{F}_{body} \Rightarrow \Psi = gz \qquad \vec{F} = -g\vec{e}_z$$

per unit mass

$$\frac{p_{atm}}{\rho} = \frac{|\vec{u}|^2}{2}\Big|_{near\ hole} - gh + \left(\frac{p_{atm}}{\rho}\right)_{near\ hole}$$

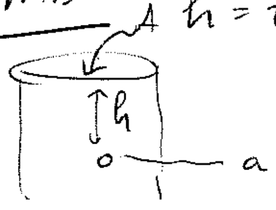
(just outside)

$$|\vec{u}|_{hole} = \sqrt{2gh}$$



(N.B.: $A\,\dot{h} = v\frac{a}{A}$, so @ $\frac{a}{A} \to \infty$

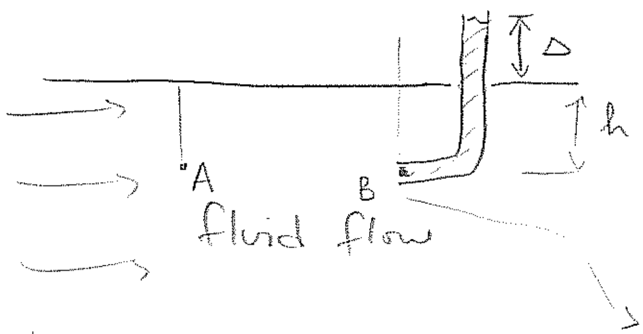
forget change of h)



Applic'n 2: Pitot tube.

e.g.

(air)



(water, sea)

$V_B = 0$

$$\frac{V_A^2}{2} + \frac{1}{\rho}p_A + \cancel{gh} = \cancel{\frac{V_B^2}{2}} + \frac{1}{\rho}p_B + \cancel{gh}$$

p_A = atm. $\qquad V_B^2/2$ = 0

"B" = stagnation point of streamline

$$V_A = \sqrt{2(p_B - p_A)}$$

here: p_A : atmospheric$/\rho$

p_B : atmospheric$/\rho$ + $g\Delta$

$$V_A = \sqrt{2g\Delta}\,,\quad \text{then}$$

element, the change of density (along streamline)

is $\Delta\rho = \dfrac{\Delta p}{\left(\frac{\partial p}{\partial \rho}\right)_s}$

but then Bernoulli's law says $\dfrac{\rho \vec{u}^2}{2} + p = \text{const}$

(ignoring Ψ & ε) so largest amount by which

p can change is $\sim \dfrac{\rho \vec{u}^2}{2}$ $\Longrightarrow$ so we <u>estimate</u>

$$\Delta p \sim \frac{\rho \vec{u}^2}{2} \Longrightarrow \Delta\rho = \frac{\Delta p}{\left(\frac{\partial p}{\partial \rho}\right)_s} \sim \frac{\rho \vec{u}^2}{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

$$\Rightarrow \frac{\Delta \rho}{\rho} \sim \frac{\vec{u}^2}{\left(\frac{\partial p}{\partial \rho}\right)_s} = \frac{u^2}{c^2}$$

$\longrightarrow$ this is a
typical argument - using ($\approx$) conservation law to
estimate 'typical' changes (trying to maximize them,
for best estimates, of
course!)

"Incompressible" $\approx \dfrac{\Delta\rho}{\rho} \ll 1 \Longrightarrow u \ll c.$

$$\text{Mach } \# = \frac{u}{c} \ll 1.$$

Now, we used $\left(\frac{\partial p}{\partial \rho}\right)_s = c^2 \rightarrow$ WHY?

"first" encounter w/ waves —

+ consider an ideal fluid

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla}\cdot\vec{u} = 0 \qquad \text{(mass conserv.)}$$

$$\rho\frac{D\vec{u}}{Dt} = \rho\vec{F} - \vec{\nabla}p \qquad \text{(momentum conserv.)}$$

+ we take $p = p(\rho, s)$, as usual

& consider an isentropic flow

recall if $s = \text{const}(\vec{x})$ at $t=0 \Rightarrow$

$\Rightarrow$ it always remains constant

so $s = s_0 \ \forall \ \vec{x}, t$ (as $Ds/Dt = 0$)

and then $Dp = \left(\frac{\partial p}{\partial \rho}\right)_s D\rho$

or $D\rho = \dfrac{1}{\left(\frac{\partial p}{\partial \rho}\right)_s} Dp = \dfrac{1}{c^2} Dp$

for now we define $c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$

$$\boxed{\begin{aligned} &\frac{1}{\rho c^2}\frac{Dp}{Dt} + \vec{\nabla}\cdot\vec{u} = 0 \\ &\rho\frac{D\vec{u}}{Dt} = \rho\vec{F} - \vec{\nabla}p \end{aligned}} \quad \boxed{**}$$

Consider a small disturbance of a fluid at rest.

The fluid at rest has p_0 & ρ_0 : $\rho_0\vec{F} = \vec{\nabla}p_0$ —

obeying equilibrium condition,
& it also has $\vec{u} = \vec{u}_0 = 0$, of course

If there's a small disturbance

$$\left.\begin{aligned} p &= p_0 + \varepsilon p_1 \\ \rho &= \rho_0 + \varepsilon \rho_1 \\ \vec{u} &= \varepsilon \vec{u}_1 \end{aligned}\right\} \quad \varepsilon \ll 1$$

let ρ_0 be uniform & so $\vec{\nabla} p_0 = \rho_0 \vec{F} = \rho_0 \vec{g} = \text{const}$

plug into eqns $\boxed{**}$, & expand to $O(\varepsilon)$:

$$\frac{1}{c^2}\left(\frac{1}{\rho_0 + \cancel{\varepsilon\rho_1}}\right)\left(\frac{\partial}{\partial t}(\varepsilon p_1) + \varepsilon \vec{u}_1 \vec{\nabla} p_0\right) + \varepsilon \vec{\nabla}\cdot\vec{u}_1 = 0$$

c^2 (evaluated @ ρ_0, p_0)

would give a term $O(\varepsilon^2)$

$$(\rho_0 + \cancel{\varepsilon\rho_1})\left(\frac{\partial}{\partial t}(\varepsilon \vec{u}_1) + \underbrace{\cancel{(\varepsilon \vec{u}_1 \cdot \vec{\nabla})\varepsilon \vec{u}_1}}_{O(\varepsilon^2)}\right) =$$

$$= \cancel{\rho_0 \vec{F}} + \varepsilon \rho_1 \vec{F} - \cancel{\vec{\nabla} p_0} - \varepsilon \vec{\nabla} p_1$$

"unperturbed" equilibrium condits —

compare terms $O(\varepsilon)$ on both sides:

$$\begin{cases} \dfrac{1}{\rho_0 c^2}\dfrac{\partial p_1}{\partial t} + \vec{\nabla}\vec{u}_1 + \dfrac{1}{c^2}\vec{u}_1 \vec{g} = 0 \\[2ex] \rho_0 \dfrac{\partial \vec{u}_1}{\partial t} = \rho_1 \vec{g} - \vec{\nabla} p_1 \end{cases}$$

to get a wave equation for p_1,
eliminate $\vec{u}_1$:

$\frac{\partial}{\partial t}$ of 1st eqn gives

$$\frac{1}{\rho_0 c^2}\frac{\partial^2 p_1}{\partial t^2} + \vec{\nabla}\dot{\vec{u}}_1 + \frac{\vec{g}}{c^2}\dot{\vec{u}}_1 = 0$$

plug $\dot{\vec{u}}_1 = \frac{\rho_1}{\rho_0}\vec{g} - \frac{\vec{\nabla} p_1}{\rho_0}$ to get,

$$\frac{1}{\rho_0 c^2}\frac{\partial^2 p_1}{\partial t^2} - \frac{1}{\rho_0}\vec{\nabla}^2 p_1 + \frac{\vec{g}}{\rho_0}\vec{\nabla}\rho_1 + \frac{\vec{g}}{c^2\rho_0}\left(\vec{g}\rho_1 - \vec{\nabla} p_1\right) = 0$$

$\times$ by ρ_0

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)p_1 = -\vec{g}\cdot\vec{\nabla}\underbrace{\left(\rho_1 - \frac{1}{c^2}p_1\right)} - \frac{g^2}{c^2}\rho_1 = (**)$$

perturbation is also adiabatic, of course so

$$\Delta\rho = \frac{1}{\left(\frac{\partial p}{\partial \rho}\right)_s}\Delta p$$
for every fluid element

$$\rho_1 - \rho_0 = \frac{1}{c^2}(p_1 - p_0)$$

or $$\rho_1 - \frac{1}{c^2}p_1 = \rho_0 - \frac{1}{c^2}p_0$$

$$\vec{\nabla}\left(\rho_1 - \frac{1}{c^2}p_1\right) = \vec{\nabla}\left(\rho_0 - \frac{1}{c^2}p_0\right) = -\frac{1}{c^2}\vec{g}\rho$$

$$= (**) = \frac{1}{c^2}\vec{g}^2(\rho_0 - \rho_1) = -\frac{g^2}{c^2}\frac{1}{c^2}(p_1 - p_0)$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)p_1 = \frac{g^2}{c^4}(p_1 - p_0)$$

<u>Now</u>: $g \sim 10\,\frac{m}{s^2}$ $\quad c \sim 300\,\frac{m}{s}$ $\Rightarrow$ $c^2/g \sim \frac{9\times 10^4}{10}\,m \sim 10^4\,m$ ((for air))

added this – no harm, $p_0 = \text{const}$

hence $$\underbrace{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)}(p_1 - p_0) = \frac{p_1 - p_0}{(10^4 m)^2}$$

l.h.s. $= 0$ solutions

have various wavelengths $\lambda \sim cT$ ← period, not temperature

so in order for r.h.s. to

matter one must have $\lambda \gg 10^4 m$

(l.h.s. evaluated on a wave w/ λ is $\sim \frac{|p_1 - p_0|}{\lambda^2} \gg \frac{|p_1-p_0|}{(10^4 m)^2}$ for $\lambda \ll 10^4 m$)

so, then we conclude $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ is the

speed of small pressure waves

($\equiv$ sound waves).

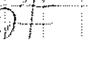
→ condition for incompressibility

is then $|\vec{u}| \ll$ speed of sound

About sound waves – much can be studied!

- ideal gas $C_{sound} = \sqrt{\gamma \frac{kT}{m}}$; $\gamma = \frac{C_p}{C_v}$

$\sim \sqrt{\frac{kT}{m}} \sim \bar{v}$.

- like EM waves: carry energy, momentum
- can study acoustics, Doppler effect
- effect of boundaries! ⟷ select wavevectors like  i.e. box
  ↳ free surface too
- radiation –  body ⟷ sound waves: dipole, quadrupole

- of liquid is non-ideal –
  – sound is attenuated
  (1st & 2nd viscosities)
  way to measure 2nd viscosity, since it's got to do w/ compressibility only

SOUND ⟹ huge !! subject, practical & fun! (DIY – !)

From now on, we'll be neglecting compressibility.

Let's study, 1st some simple examples of viscous, incompressible flows: For such flows, we have a simpler set of (see p. 77) equations:

most general eqns

- mass conservation + incompressibility $\Leftrightarrow$ $\vec{\nabla} \cdot \vec{v} = 0$
- N.S. eqn:

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{F} - \vec{\nabla} p + \mu \Delta \vec{u}$$

(Laplacian, not "change of")

- entropy

$$\rho T \frac{Ds}{Dt} = \varkappa_T \Delta T + 2\mu (e_{ij})^2$$

- of course, equation of state

→ to simplify things even further, recall remark on

p. 79, bottom (C — conditions involving $\mu, \varkappa_T, \ldots$) (for incompressibility)

back to $Dp = \left(\frac{\partial p}{\partial \rho}\right)_s D\rho + \left(\frac{\partial p}{\partial s}\right)_\rho Ds$, using (ρ, s) as parameters of state

so $$\frac{D\rho}{Dt} = \underbrace{\frac{1}{c^2}\frac{Dp}{Dt}} - \frac{1}{c^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{Ds}{Dt}$$

change of density due to Δp or due to Δs (dissipation)

* we studied, so far, condition that this term be small — and argued that $U/c \ll 1$ must be met (to ensure $\Delta\rho/\rho \ll 1$)

Now, more generally, in a given flow we imagine a typical U & a typical length scale L (on which U as well as other characteristics of flow vary significantly). So, spatial derivatives of U are $\sim \frac{U}{L}$

& $\bar{\nabla}\cdot\bar{U} \approx 0$ means $|\nabla\cdot\bar{U}| \ll \frac{U}{L}$.

From continuity equation $\dot{\rho} + \vec{\nabla}(\vec{u}\rho) = 0$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla}\cdot\vec{u} = 0$$

we have than $\left|\frac{1}{\rho}\frac{D\rho}{Dt}\right| \ll \frac{U}{L}$ $\Rightarrow$ we must then (plugging above $\frac{D\rho}{Dt}$)

ensure that $$\left|\frac{1}{c^2\rho}\frac{Dp}{Dt} - \frac{1}{\rho c^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{Ds}{Dt}\right| \ll \frac{U}{L}$$

require each term $\ll \frac{U}{L}$ $\rightarrow$

$$\rightarrow \left| \frac{1}{\rho c^2} \left(\frac{\partial p}{\partial s}\right)_\rho \frac{Ds}{Dt} \right| \ll \frac{U}{L}$$

TD relations

but
$$\frac{1}{\rho c^2}\left(\frac{\partial p}{\partial s}\right)_\rho = -\frac{1}{\rho c^2}\left(\frac{\partial p}{\partial \rho}\right)_s\left(\frac{\partial \rho}{\partial s}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \frac{1}{\left(\frac{\partial s}{\partial T}\right)_p} =$$

$$= \frac{\beta T}{c_p}\ ; \quad \beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p$$

used $\left(\frac{\partial p}{\partial s}\right)_\rho \left(\frac{\partial \rho}{\partial p}\right)_s \left(\frac{\partial s}{\partial \rho}\right)_p = -1$

(derivatives)
can explicitly do ✓ for ideal gas if you don't trust general relation $\rightarrow \beta \sim \frac{1}{T}$

$\frac{\beta U}{c_p}\frac{\nu}{L} \sim \frac{m}{kT}\frac{U}{L}\nu$; $c_p \sim \frac{k}{m}$

about right!

air

$\longrightarrow$ plug $\frac{Ds}{Dt}$ from p (77)

$\Downarrow$

$$\begin{cases} \beta \Delta T \dfrac{\varkappa_T}{LU} \ll 1 \\[2ex] \dfrac{\beta U^2}{c_p}\dfrac{\mu}{\rho L U} \ll 1 \end{cases}$$

$\Longrightarrow$ $L \sim 1\,cm$
$U \sim 10\,cm/s$
$\Delta T \sim 10^\circ C$

	$\beta \Delta T \frac{\varkappa_T}{LU}$	$\frac{\beta U^2}{c_p}\frac{\mu}{\rho L U}$
air	10^{-3}	10^{-9}
water	10^{-6}	10^{-17}

on the other hand if $\Delta T \sim 100^\circ C$ & $LU \sim .1\,cm^2/s$ $\rightarrow$ relevant

in words: changes of density due to dissipative heating are not important if these conditions hold

under such circumstances, ignore changes of entropy of fluid elements (as we ignore changes of density and these are directly related).
Hence, energy balance (entropy change) eqn. – irrelevant (simple!)