$\boxed{\text{Warning}}$ → while we'll be ignoring $\Delta\rho$ & Δs, in real life there may be situations where this can't be done — so you'll need (then!) to brush up on thermodynamics!

"

So we disposed of all but:

$$\begin{cases} \vec{\nabla}\cdot\vec{u} = 0 \\ \rho \dfrac{D\vec{u}}{Dt} = \rho\vec{F} - \vec{\nabla}p + \mu\,\Delta\vec{u} \end{cases}$$

→ since $\rho = \text{const}$ there are now 4 eqns for 4 va

(assume uniform, to boot)

(μ is assumed uniform, too)

For ρ - uniform, we now have

$$\vec{\nabla}\cdot\vec{u} = 0 \quad , \quad \frac{D\vec{u}}{Dt} = \vec{F} - \frac{1}{\rho}\vec{\nabla}p + \nu\,\Delta\vec{u}$$

recall $\nu_{air} \sim .15\ \frac{cm^2}{s}$

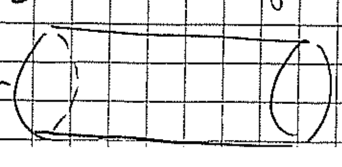
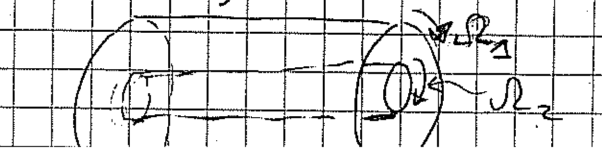
$\nu_{water} \sim .01\ \frac{cm^2}{s}$

(@ 15°C)

Now there are still bad

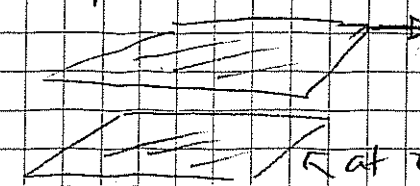
$$\frac{D\bar{u}}{Dt} = \dot{\bar{u}} + (\bar{u}\cdot\bar{\nabla})\bar{u}$$

nonlinear: there's clearly a way to "avoid", by considering a "1-d" flow which is also steady, e.g.:

Poiseuille (1840)  or Couette 

or, simpler even: moves w/ $\vec{v}$

two parallel plates



in this case, clearly, fluid will move w/ $\vec{u} \parallel \vec{v}$

(imagine we started @ rest, increasing $\vec{v}$ from 0, and reaching a steady state)

since $\vec{u}$ will only vary as a f-n of distance between plates, also clear that

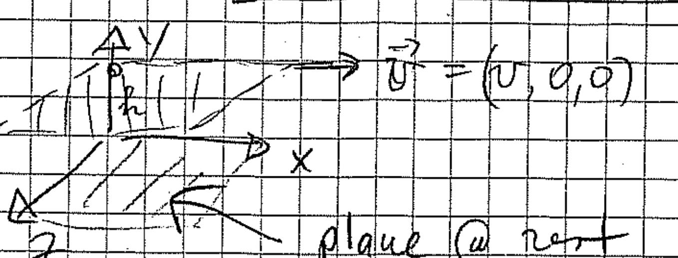
$$(\vec{u}\nabla)\vec{u} = 0$$, so eqn. linearizes

Same linearization occurs in flow down a pipe (Poiseulle), as $\vec{u}$ varies in radial direction but is always $\perp$ to it [unless an instability occurs — later!], so again $(\vec{u}\nabla)\vec{u} = 0$.

Of course the same holds for Couette flow, too [modulo same remark on instability!]

These three flows are "simple" — but a more careful consideration reveals a lot of "stuff" $\longrightarrow$ instabilities & turbulence.

OK, now LET'S SOLVE SOME EQNS!



$\vec{u} = (u, 0, 0)$ — along x

$u = u(y)$ — only dep on y

$\left.\begin{array}{l} u(0) = 0 \\ u(h) = v \end{array}\right\}$ fluid sticks to boundaries

All quantities will depend on y only – 9B

– so $\bar{\nabla}\cdot\bar{v} = 0$ since v_x is x-indep. & $v_z = v_y = 0$.

& N.S. is ($\bar{F} = 0$, too)

f-n of y, not x

must be constant

$$\nu \Delta \vec{u} = \frac{1}{\rho}\vec{\nabla} p \Rightarrow \nu \frac{d^2}{dy^2} u = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\left.\begin{aligned} 0 &= \frac{1}{\rho}\frac{\partial p}{\partial y} \\ 0 &= \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \text{p indep on } (y, z)$$

so if $\frac{dp}{dx} \neq 0$ flow is

due to moving plates & pressure gradient

let $p = p_0 - p_1 x$, linear f-n of x, then $u'' = -\frac{1}{\mu} p_1$

Solution is $u(y) = -\frac{1}{2\mu} p_1 y^2 + by + c$

$$u(0) = 0 \Rightarrow c = 0$$

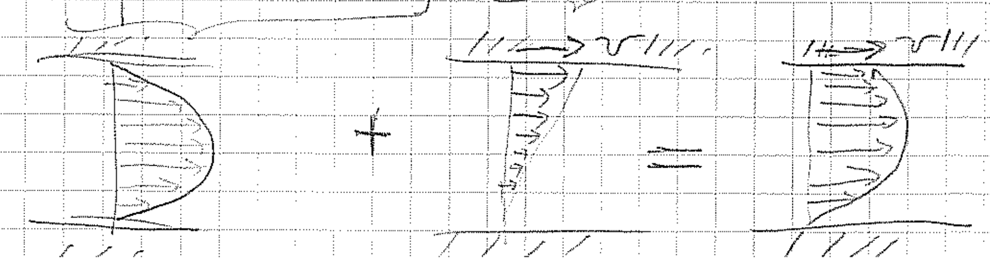
$$u(h) = v \Rightarrow bh - \frac{1}{2\nu\rho} p_1 h^2 = v$$

$$b = \frac{v}{h} + \frac{1}{2\nu\rho} p_1 h$$

and we have

$$u(y) = -\frac{1}{2\nu\rho} p_1 y^2 + \frac{1}{2\nu\rho} p_1 h y + \frac{v}{h} y$$

$$u(y) = \underbrace{\frac{1}{2\nu\rho} p_1 y(h - y)} + \underbrace{\frac{v}{h} y}$$



$$\sigma_{xy} = \hat{\sigma}_{xy} = 2\mu\, e_{xy} = 2\mu\,\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \mu\left(\frac{\partial u_x}{\partial y} + \cancel{\frac{\partial u_y}{\partial x}}\right)$$

the only other $\neq 0$ components are $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p = -(p_0 - p_1 x)$

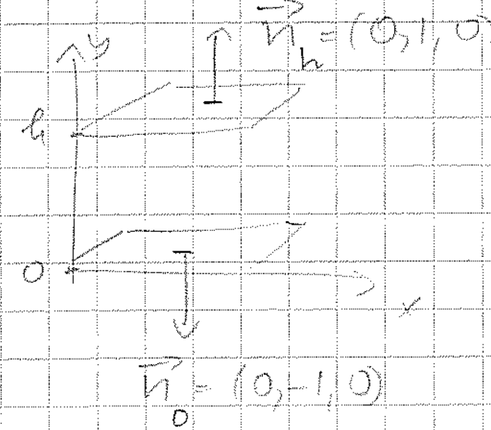
so

$$\sigma_{xy} = \mu \frac{\partial u_x}{\partial y} = \mu \frac{\partial}{\partial y}\left(\frac{1}{2\mu}\, p_1 (h y - y^2) + \frac{v}{h} y\right)$$

$$\sigma_{xy}(y) = \frac{p_1 h}{2} - p_1 y + \frac{\mu v}{h}$$

so $\sigma_{xy}(0) = \frac{1}{2} p_1 h + \frac{\mu v}{h}$

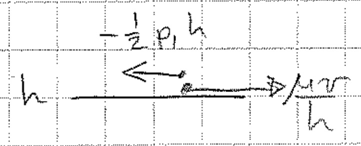
$\sigma_{xy}(h) = -\frac{1}{2} p_1 h + \frac{\mu v}{h}$



@ $y = h$ $\quad f_x(h) = \sigma_{xy}(h)\, n_h^y$

$= +\sigma_{xy}(h)$

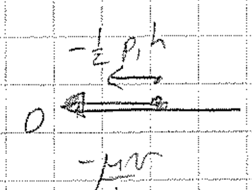
$= -\frac{1}{2} p_1 h + \frac{\mu v}{h}$



@ $y = 0$ $\quad f_x(0) = \sigma_{xy}(0)\, n_0^y$

$= -\sigma_{xy}(0)$

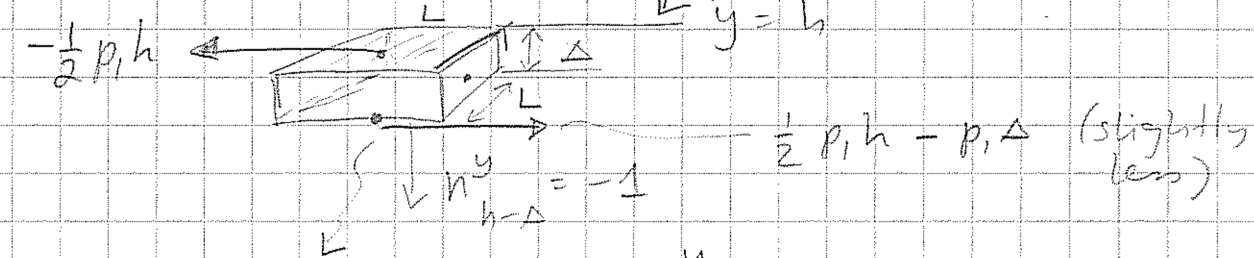
$= -\frac{1}{2} p_1 h - \frac{\mu v}{h}$



• $\frac{\mu v}{h}$ is friction force due to relative motion of walls

What about the other force?

let $v=0$ for simplicity

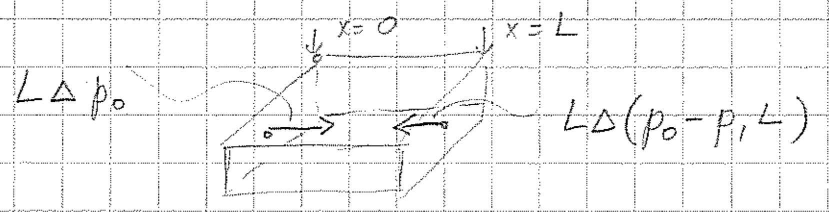


$$f_x = \sigma_{xy}(y = h-\Delta)\cdot n^y_{h-\Delta} = -\sigma_{xy}(h-\Delta)$$

$$= -\left(\frac{p_1 h}{2} - p_1(h-\Delta)\right)$$

$$= -\left(\frac{p_1 h}{2} - p_1 h + p_1\Delta\right) = -\left(-\frac{p_1 h}{2} + p_1\Delta\right)$$

$$= +\frac{1}{2}p_1 h - p_1\Delta$$



total $\longrightarrow L^2\Delta p_1$ (pressure gradient)

$$\underbrace{\left(-\frac{1}{2}p_1 h - \left(\frac{1}{2}p_1 h - p_1\Delta\right)\right)L^2 =}$$

$$= \underbrace{p_1\Delta L^2}_{\text{balance fine!}}$$

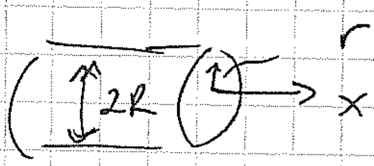
Why we "forget" energy equation (usually) while solving $\bar{\nabla}\cdot\bar{u} = 0$

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{F} - \bar{\nabla} p + \mu \Delta \bar{u} \quad ?$$

$\rightarrow$ $\frac{D\varepsilon}{Dt} = 2\nu (e_{ij} e_{ij})^2 + \alpha_T$ — contribution

- equation is certainly there ...
- friction increases internal energy ... can eventually cause significant heating; let's estimate how much ?

pipe of radius R: $u_x \sim \frac{p_1}{\nu\rho}(R^2 - r^2)$



$$\frac{\partial u_x}{\partial r} \sim \frac{p_1 R}{\nu\rho} \sim \text{"}e_{ij}\text{"}$$

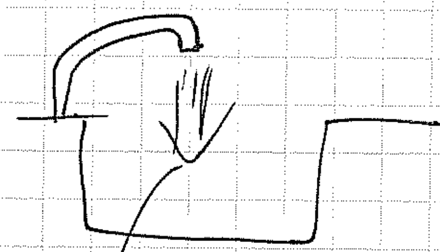
$$\frac{D\varepsilon}{Dt} \sim \nu\left(\frac{\partial u_x}{\partial r}\right)^2 \sim \nu \frac{p_1^2 R^2}{\nu^2\rho^2} \sim \frac{p_1^2 R^2}{\nu\rho^2}$$

average $\bar{u}_x \sim \frac{p_1 R^2}{\nu\rho}$ so $p_1 \sim \frac{u_x \nu\rho}{R^2}$

so $\frac{D\varepsilon}{Dt} \sim \frac{p_1^2 R^2}{\nu\rho^2} \sim \frac{u_x^2 \nu^2 \rho^2}{\nu\rho^2 R^2} \sim \bar{u}_x^2 \nu$

$$\frac{D(\varepsilon\rho)}{Dt} \sim \bar{u}_x^2 \, \nu \rho$$

increase of internal energy per unit volume ($\varepsilon\rho$) in unit time



kitchen sink

$r \sim .5$ cm

1 l fills in ~ 13 s

hence $\bar{u} \sim 1 \frac{cm}{s}$

$\nu \sim .01 \frac{cm^2}{s}$

$\rho \sim 1 \frac{g}{cm^3}$

→ 10,

per 1s, we have that $1 cm^3$ of water gets $.01 \frac{g \times cm^2}{s^2}$ of energy

$$= .01 \frac{10^{-3} kg \; 10^{-4} m^2}{s^2} \sim 10^{-9} J$$

$1 cm^3 \simeq 1g$, so per second $1g$ gets 10^{-9} J

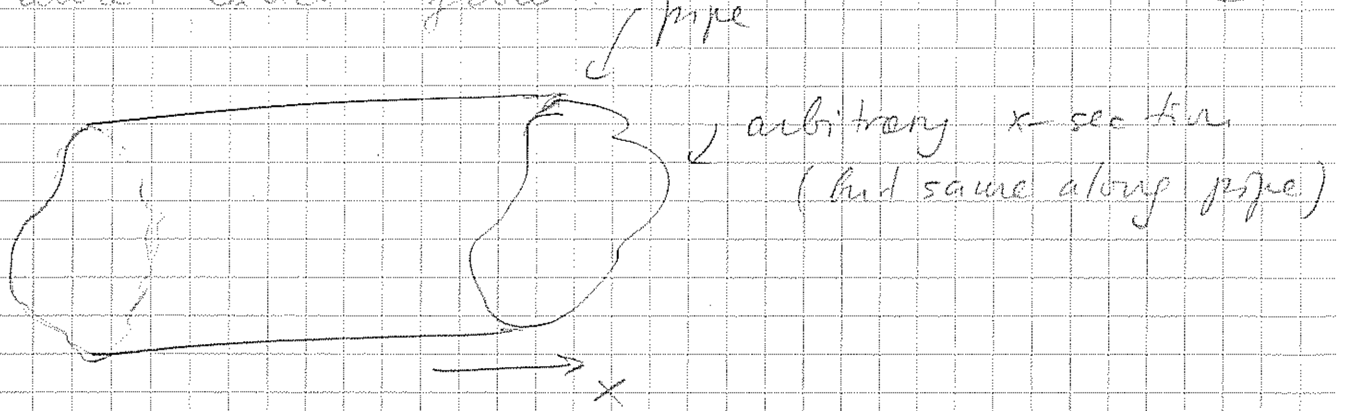
--→ to raise T by 1K need 4 J (per 1g)

--→ hence will happen in $\sim 10^9 - 10^{10}$ s ~

$\sim 10^2 \div 10^3$ yrs ($1yr \approx 10^7 s$)

(effect of $T_1 - T_0$ in $\mathring{\varepsilon}$ -- use $\varkappa_T = .6 \frac{W}{mole \cdot kg}$, 1 mole $= (H_2O) = 18g$)

One more exact flow:



$$\vec{u} = (u, 0, 0)$$

mass conserv.: $\vec{\nabla}\cdot\vec{u} = 0 \Rightarrow \partial_x u = 0$ — clearly true

momentum eqn: $u = u(y,z)$ only, $(\vec{u}\cdot\vec{\nabla})\vec{u} = 0$
by symmetry

$$\rho \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} p + \mu \Delta \vec{u}$$

y, z components: $0 = -\partial_y p + 0 \Rightarrow p = p(x)$

$$0 = -\partial_z p + 0$$

x-component: $\rho \dfrac{\partial u}{\partial t} = -\dfrac{\partial p}{\partial x} + \mu\left(\partial_z^2 + \partial_y^2\right) u$

since $u = u(y,z)$ must be $\dfrac{\partial p}{\partial x} = $ constant

hence for a stationary flow along the pipe

$$\nu\left(\partial_y^2 + \partial_z^2\right) u(y,z) = +\left(\frac{\partial p}{\partial x}\right)\frac{1}{\rho} = \text{const}$$

$p = p_0 - p_1 x$ as before

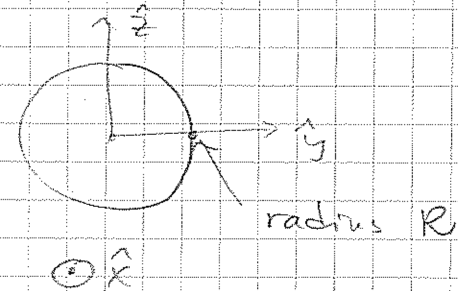
$$\nu\left(\partial_y^2 + \partial_z^2\right) u(y,z) = -p_1 \frac{1}{\rho} \quad (p_1 = \text{const})$$

$$\to \quad \Delta_{(2)} u = -\left(\frac{p_1}{\nu\rho}\right) = \text{const}$$

2d Laplacian in (y, z)

$$+ \text{ b.c. } \quad u\Big|_{y,z \,\in\, \text{surface of pipe}} = 0.$$

If x-section is a circle



$$\Delta_{(2)} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2}$$

but we have cylindrical symmetry so

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{d}{dr} u\right) = -\frac{p_1}{\nu\rho}$$

solution is clearly $\to$ $u(r) = -\dfrac{p_1}{4\nu\rho} r^2 + a \ln r + b$

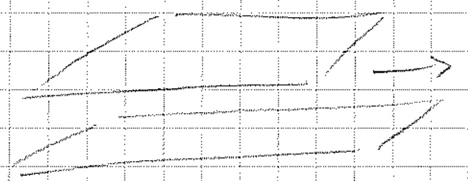
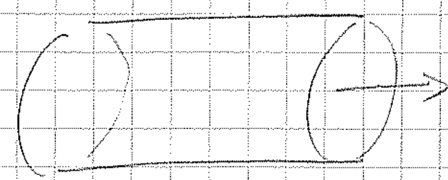
$$u(R) = 0 = -\frac{p_1}{4\nu\rho} R^2 + a \ln R + b$$

+ regularity @ $r = 0$ (demands $a = 0$)

((recall $\Delta_{(2)} \ln|\vec{r}| = 2\pi\, \delta^{(2)}(\vec{r})$,
i.e. $\ln|\vec{r}|$ is the Green's f-n of 2d Laplace operator))

so $b = \dfrac{p_1}{4\nu\rho} R^2$ & $u(r) = \dfrac{p_1}{4\nu\rho}\left(R^2 - r^2\right)$

So the character of the flow is same for

 or 

$\Rightarrow$ a parabola.

the flow thru the x section of the pipe
(mass)
(amount of liquid thru pipe in unit time)

is $$\int_0^R \underbrace{\rho u}_{\text{flux of mass}} 2\pi r\, dr = \int_0^R \rho\, 2\pi r (R^2 - r^2)\frac{p_1}{4\nu\rho}\, dr$$

$$= \frac{\rho\, p_1}{4\nu\rho}\, 2\pi \left(R^2 \frac{R^2}{2} - \frac{R^4}{4}\right)$$

$$= \frac{p_1 \rho}{4\nu\rho}\, 2\pi \frac{R^4}{4} = \frac{\pi}{8}\frac{p_1}{\nu} R^4$$

sensible: $\sim |\vec{\nabla} p|$ empirically 1st

$\sim \frac{1}{\nu}$ Poiseuille

Hagen ~1840

$\sim \rho$ & R^4 formula-Stokes 1845

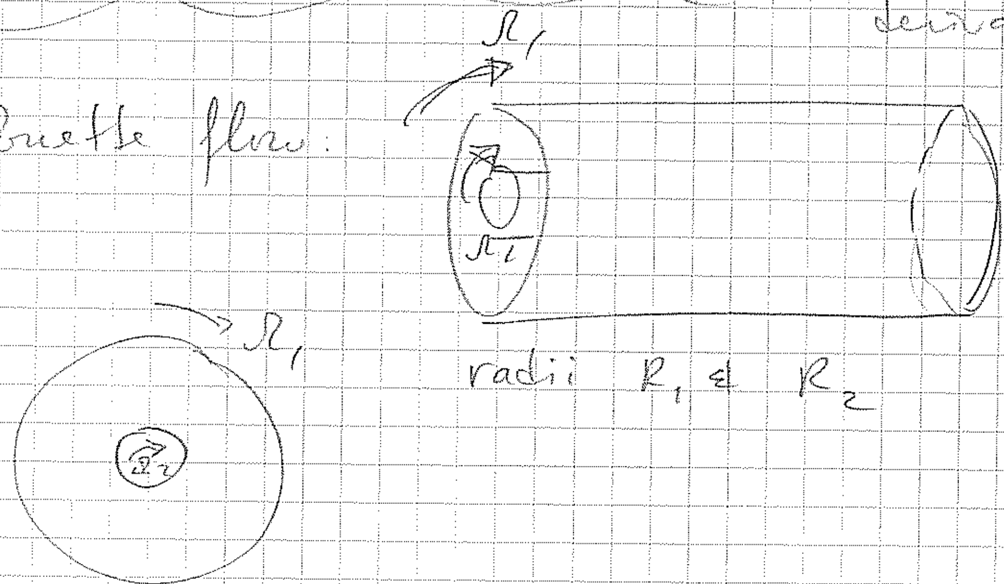
(~ 2nd power of x-sectional area, not 1st.)

→ another observ: all soltns so far ~ $\frac{1}{\nu}$!

* singular as $\nu \to 0$

* mathematically — ν multiplies highest derivative in eq^n

Couette flow:



radii R_1 & R_2

h.w.

Another note: * soltns where no external $\vec{\nabla} p$ is present are indep of ν

* soltns in flow driven by $\vec{\nabla} p$ are $\sim \frac{1}{\nu}$

(for an ideal liquid w/ $\vec{\nabla} p$ & no ext. force soltn would be singular → as it'd accelerate forever, w/ no counterbalance for $\vec{\nabla} p$)

Next = Reynolds # & similarity

for stationary (incompressible flow) we have N.-S. eq^n

$$\rho (\vec{u}\vec{\nabla})\vec{u} = \rho \vec{F} - \vec{\nabla} p + \mu \Delta \vec{u}, \quad \vec{\nabla} \cdot \vec{u} = 0.$$