Fun (& somewhat relevant - e.g. HW 3) stuff

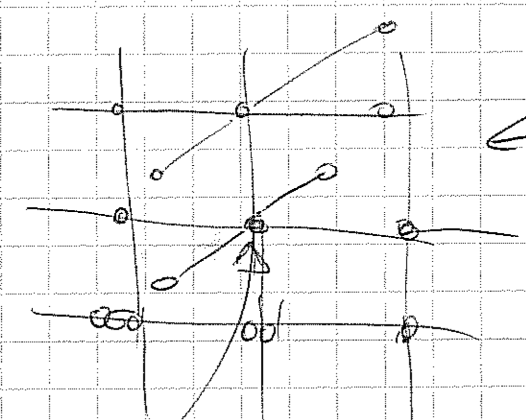
- boundary conditions on fluid interfaces

- We only did fluid/solid : $\vec{v}_{fluid}\Big|_{\substack{at\\solid}} = 0$

"no slip"

- But much other interesting stuff -
  - fluid on fluid say - ones that do not mix ...

→ surface tension ←



← ∞ solid (3d!)

+ everybody has 6 neighbors

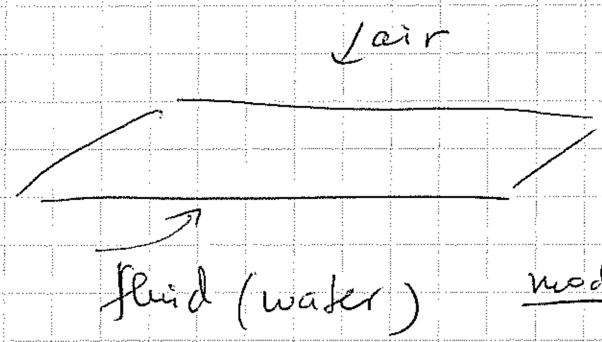
+ molecules on surface only have 5 neighbors

↓ they are bound less than those inside -

- so they have an excess positive energy

binding E of molecule inside : ϵ

excess positive energy of molecules on surface = $\frac{1}{6}\epsilon$



model: molecular size $\sim L_{molecule}$

density of H_2O $1\,\frac{g}{cm^3}$

mass of H_2O $\sim 3\times 10^{-23}\,g$

(18 u)

($\equiv$ atomic mass units)

vaporization heat

$\sim 2000\,\frac{J}{g}$

$\left(> 4\,\frac{J}{Kg} \times 100\,K ! \right)$

binding energy ?? $\epsilon \sim (\text{mass of molecule}) \times \begin{pmatrix}\text{vaporization}\\ \text{heat}\end{pmatrix}$

$$\sim 3\times 10^{-23}\,g \times 2000\,\frac{J}{g}$$

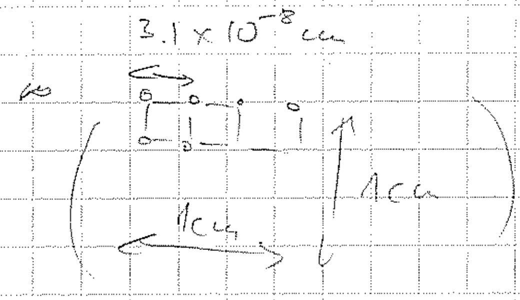
$$\sim 6\times 10^{-20}\,J$$

typical distance ? between molecules =

$$= \sqrt[3]{\frac{\text{volume}}{\text{molecule}}} \sim \sqrt[3]{\frac{\text{mass of } H_2O}{\text{density}}} \sim \sqrt[3]{\frac{3\times 10^{-23}\,g}{1\,g}\,cm^3}$$

$$\sim \left(3\times 10^{-23}\right)^{1/3} cm \sim 3.1\times 10^{-8}\,cm$$

$$\left(\frac{\text{surface energy}}{\text{unit area}}\right) \sim \text{"}\frac{\epsilon}{6}\text{"} \times \frac{\#\ \text{molecules}}{cm^2} \sim (*)$$



$$\Rightarrow \frac{\#\text{ molecules}}{\text{cm}^2} \sim \frac{1}{(3.1)^2}\frac{10^{16}}{\text{cm}^2} \sim \frac{10^{15}}{\text{cm}^2}$$

"surface tension"

$$(*)\ \underline{\alpha} \sim \frac{\text{surface energy}}{\text{unit area}} \sim \frac{6\times 10^{-20}\,\text{J}}{6} \times \frac{10^{15}}{\text{cm}^2} \sim 10^{-5}\,\frac{\text{J}}{\text{cm}^2}$$

$$\left[\ \underline{\text{measured}}\ 0.072\,\frac{\text{J}}{\text{m}^2} \sim \frac{0.07}{10^4}\,\frac{\text{J}}{\text{cm}^2} \sim 7\times 10^{-6}\,\frac{\text{J}}{\text{cm}^2}\ \right]$$

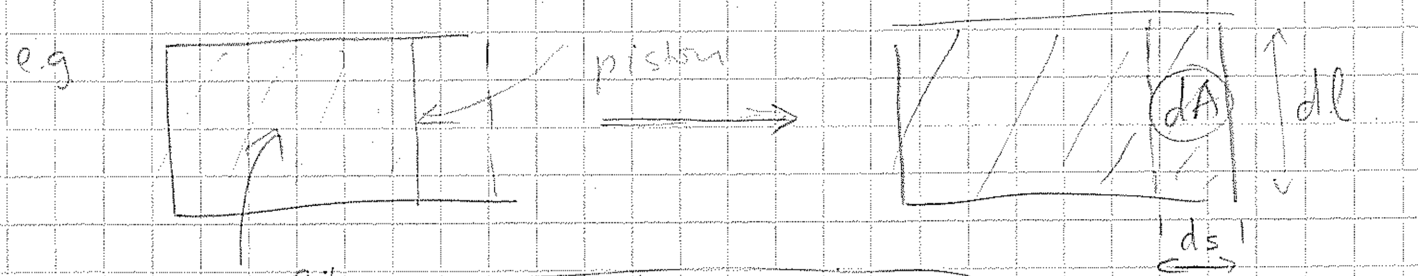
@ 25 °C

NOT BAD!!

This means that to increase surface area by 1 cm² need 10^{-5} J. (not much, indeed!)

This surface energy can be thought of as due to "surface tension":

e.g.



energy to increase area $= \alpha\, dA$

$\Rightarrow$ Work done to move "piston"

((this is a 1d piston, not 2d as in pressure))

$$dW = \alpha \, dA$$

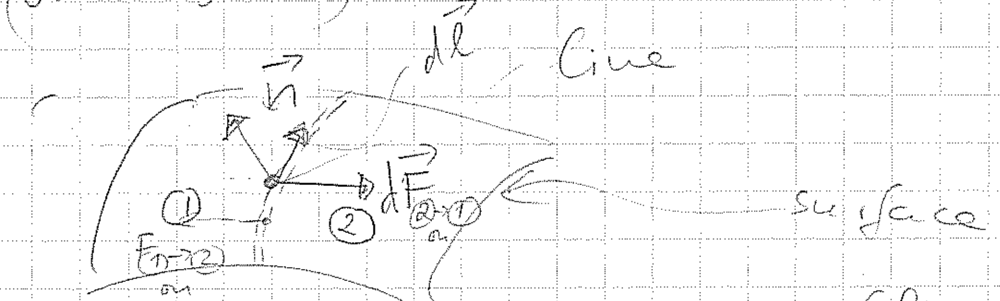
need to apply force $dF \cdot ds = dW = \alpha \, dA$

$$\Rightarrow dF = \alpha \frac{dA}{ds} = \alpha \, dl$$

$$dA = dl \, ds.$$

$\Rightarrow$ surface tension is associated with a force acting on a line ("line forces" vs "surface forces")

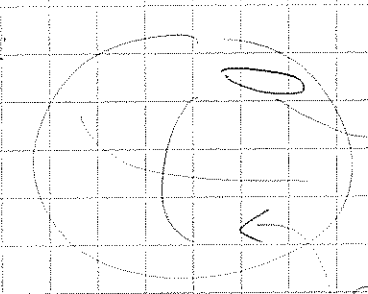
$$\frac{(\text{Force})}{(\text{unit length})} = (\text{surface tension}) = \frac{(\text{surface energy})}{(\text{unit area})}$$

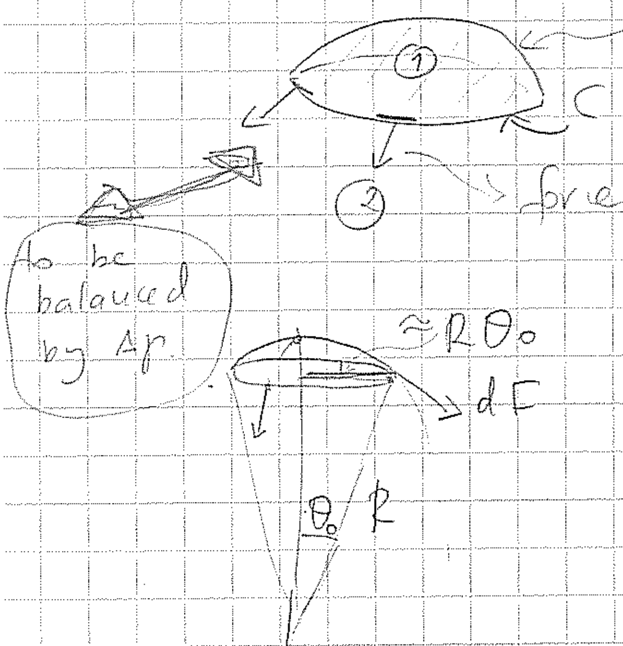


$$d\vec{F} = \alpha \, d\vec{l} \times \vec{n} = \begin{pmatrix} \text{linear (i.e. on } dl\text{)} \\ \text{Force that (2)} \\ \text{exerts on interface (line)} \\ \text{w/ (1)} \end{pmatrix}$$

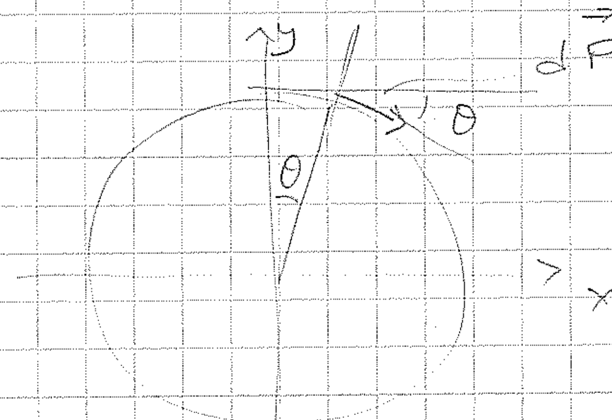
sign is determined by common sense – think of surface as membrane, the tendency is to shrink – so force of (2) on portion should be in (as $\alpha > 0$)

Ex:





$$d\vec{F} = \alpha\, d\vec{l}$$



$\theta_0 \ll 1$

$$(dF)_y = -\alpha\, dl \sin\theta_0 \approx -\alpha\, dl\, \theta_0$$

$$\oint dl = 2\pi R \theta_0 \quad (\theta \text{ small})$$

$$F_y(\text{on } C) = -\alpha\, 2\pi R\, \theta_0^2$$

what compensates it?

↓

pressure inside is larger!

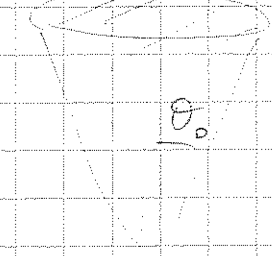
by how much?

$$\Delta p = (p_{in} - p_{out}) \qquad \Delta p \times (\text{Area of } C) = F_y(\text{on } C)$$

what is the area of C?

area of sphere $= 4\pi R^2 = (\text{solid angle}) \times R^2$

solid angle of



θ_0

$$= 2\pi \int_0^{\theta_0} \sin\theta \, d\theta \qquad \left(\int_0^{\pi} \sin\theta \, d\theta = -\cos\theta \Big|_0^{\pi} = 2 \right)$$

if θ_0 small

$$\simeq 2\pi \int_0^{\theta_0} \theta \, d\theta =$$

$$= \frac{2\pi\, \theta_0^2}{2} = \pi \theta_0^2 \quad \text{so area is } \pi \theta_0^2 R^2$$

so $\Delta p \times \pi \theta_0^2 R^2 = \alpha\, 2\pi R\, \theta_0^2$

$\Delta p = 0$ $R \to \infty$

ignore? $\Delta p_g \sim \rho g L$ 2.7 mm @ 25°C

$\Delta p_\alpha \sim \frac{\alpha}{L}$ $L \ll \sqrt{\frac{\alpha}{\rho g}}$

$$\Rightarrow \Delta p = \frac{2\alpha}{R} \longrightarrow (\text{Young-Laplace})$$

- ignore g
- $L \gg \sqrt{\frac{\alpha}{\rho g}}$ ignore α

Same thing from energy: increase radius by dR

work to increase radius by R → surface tension → against Δp. ⟶ $dW = \alpha\, dA - \Delta p\, dV$

$\|$

0 in equilibrium should be stable, so no collapse or expand

$dA = d(4\pi R^2) = 8\pi R\, dR$

$dV = d(\frac{4}{3}\pi R^3) = 4\pi R^2 dR$

$\rightarrow \alpha\, dA = \Delta p\, dV \Rightarrow 8\pi R \alpha = 4\pi R^2 \Delta p$

⟶ same.

$\rho = 1\, g/cm^3$

$g = 10^3 \frac{cm}{s^2}$

$\alpha \sim 10^{-5} \frac{J}{cm^2}$

$1J = kg \frac{m^2}{s^2}$

$L_c \sim \sqrt{\frac{10^{-5} 10^3 10^4}{10^3}} \sim .1\, cm$

not bad

Why all this? — b.c.!!

We had, before:

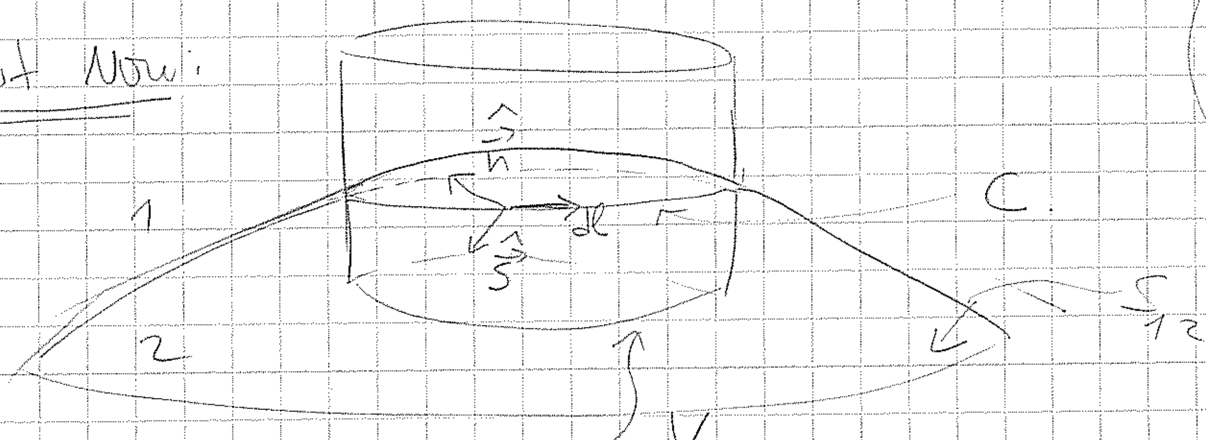
$$\int_V \rho \frac{Du^i}{Dt}\, dV = \int_V F^i\, dV + \oint_{\partial V = S} \sigma^{ij}\, dS^j$$

either that of 1 or 2

momentum V — body force

$$+ \int_C s^i{}_\alpha\, dl.$$

But Now:



momentum changes due to forces $\leftrightarrow$ add line force!

force acting on line C due to outside stuff

(as per our example!)

$\hat{\vec{s}}$ is tangent to S_{12}

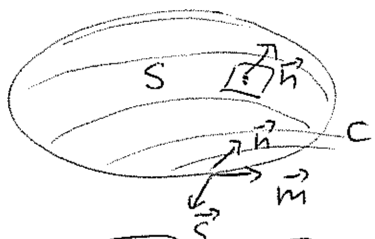
$d\vec{l}$ is also tangent to S_{12}, but also to C.

$\hat{\vec{n}}$ is normal to S_{12} (not $S = \partial V$)

Some math ...

$$(*) \quad \int_{C=\partial S} \vec{F}\cdot\vec{m}\, dl = \int_S dS\, \vec{n}\cdot(\vec\nabla\times\vec F)$$

this is $\oint \vec A\cdot d\vec l = \int \vec B \cdot d\vec S$ from E&M $\nearrow$ $= \int \vec\nabla\times\vec A\cdot d\vec S$



$\boxed{\vec m\times\vec n = \vec s}$

$\vec m$ = tangent to C
$\vec n$ = normal to S

let $\vec F = \vec f\times\vec b$, $\vec b$ = const

$$(1): \quad \vec\nabla\times(\vec f\times\vec b) = -\vec b\,(\vec\nabla\cdot\vec f) + (\vec b\cdot\vec\nabla)\,\vec f \quad (\text{for constant } \vec b)$$

also
$$(\vec f\times\vec b)\cdot\vec m = -(\vec f\times\vec m)\cdot\vec b \quad ;(2)$$

$$(*) \quad \int_C \vec F\cdot\vec m\, dl = \int_C (\vec f\times\vec b)\cdot\vec m\, dl \overset{(2)}{=} -\vec b\cdot\int_C \vec f\times\vec m\, dl$$

$$\| $$

$$\int_S dS\, \vec n\cdot(\vec\nabla\times(\vec f\times\vec b)) \overset{(1)}{=} \int_S dS\, \vec n\,\overbrace{(\vec b\cdot\vec\nabla)\vec f} - \int_S dS\,(\vec n\cdot\vec b)(\vec\nabla\cdot\vec f)$$

so we have, since $\vec b$ is arbitrary

$$\int_C \vec f\times\vec m\, dl = \int_S dS\,\vec n\,(\vec\nabla\cdot\vec f) - \int_S dS\,(\vec\nabla f^i)\, n^i \quad (**)$$

now let $\vec f = \alpha\vec n \Rightarrow \vec f\times\vec m = \alpha\underbrace{\vec n\times\vec m}_{-\vec s} = -\alpha\vec s$

$(**)$:

$$-\int_C \alpha\vec s\, dl = \int_S dS\,\vec n\,(\vec\nabla\cdot\underbrace{(\alpha\vec n)}_{\vec f}) - \int_S dS\,\vec\nabla\underbrace{(\alpha n^i)}_{\vec f}\, n^i =$$

$$= \int_S dS\Big[\vec{n}\,(\underbrace{\partial_i \alpha \cdot n_i}_{} + \alpha\,\partial_i n_i) - \vec{\nabla}\alpha\,\underbrace{n^i n^i}_{1} - \alpha\,\underbrace{\vec{\nabla} n^i\, n^i}_{=0}\Big]$$

α is a property of the surface - it changes only // S

$\Rightarrow \vec{\nabla}\alpha \perp \vec{n} \Leftrightarrow \partial_i \alpha \cdot n^i = 0$

$$\vec{\nabla}(\vec{n}^2) = \vec{\nabla} 1 = 0$$
$$\parallel$$
$$2 n_i \vec{\nabla} n_i$$

$$= \int_S dS\big(\vec{n}\,\alpha(\vec{\nabla}\cdot\vec{n}) - \vec{\nabla}\alpha\big)$$

$$\Rightarrow \int_C \alpha\,\vec{s}\,dl = \int_S dS\big(\underbrace{\vec{\nabla}\alpha}_{} - \alpha\,\vec{n}\,(\vec{\nabla}\cdot\vec{n})\big)$$

$\vec{\nabla}\alpha$ is $\perp \vec{n}$, as already mentioned

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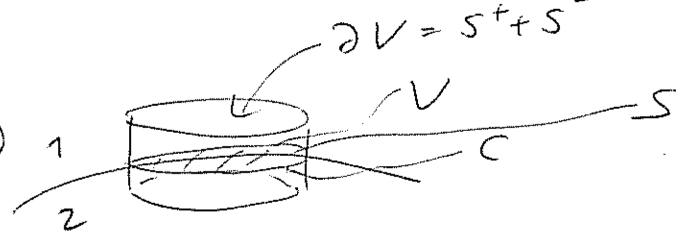
then back to p. 7



$$\underbrace{\int_V \rho \frac{Du^i}{Dt}\,dV = \int_V F^i\,dV}_{} + \underbrace{\oint_{\partial V} dS^j \sigma^{ij} + \int_C s^i \alpha\,dl}_{}$$

for small "beer can" these vanish

these should balance out

$s_j \in \partial S$

$$\oint_{\partial V} dS^j \sigma^{ij} + \int dS\big(\vec{\nabla}\alpha - \alpha\,\vec{n}(\vec{\nabla}\cdot\vec{n})\big)$$

$\partial V = S^+ + S^-$

ignore side

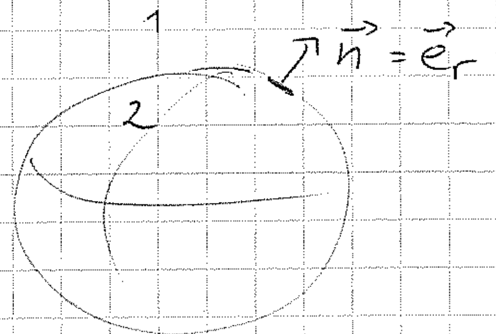
$$\oint_S dS\Big[n^j\big(\underset{\uparrow S^+}{\sigma_1^{ij}} - \underset{\uparrow S^-}{\sigma_2^{ij}}\big) + \partial^i \alpha - \alpha\,n^i(\vec{\nabla}\cdot\vec{n})\Big] = 0.$$

$$\left(\sigma_1^{ij} - \sigma_2^{ij}\right) n^j + \partial^i \alpha - \alpha n^i (\vec{\nabla}\cdot\vec{n}) = 0$$

apply to: $\alpha = \text{const}$

1. $\sigma^{ij} = -p\,\delta^{ij}$, $\mu = 0$

spherical surface



$$(-p_1 + p_2)\, n^i - \alpha n^i (\vec{\nabla}\cdot\vec{n}) = 0$$

$$(p_2 - p_1) = \alpha\, \vec{\nabla}\cdot\vec{n}$$

p. 124 $\vec{\nabla}\cdot\vec{F}$ with $F_r = 1$, $F_\theta = F_\varphi = 0$

$$\vec{\nabla}\cdot\vec{n} = \vec{\nabla}\cdot\vec{e}_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = \frac{2}{R}$$

$$p_2 - p_1 = 2\frac{\alpha}{R}$$

(works, as it should!)

BTW. other applic: tree sap ↑ 100 meters!!

2. ignore α --- but let there be μ

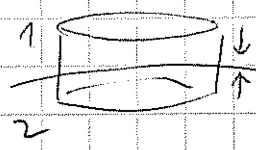
$$\sigma^{ij} = -p\,\delta^{ij} + \mu\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right)$$

(4)

$$\boxed{(-p_1 + p_2)\, n^i + \mu_1\left(\frac{\partial u_1^i}{\partial x^j} + \frac{\partial u_1^j}{\partial x^i}\right) n^j - \mu_2\left(\frac{\partial u_2^i}{\partial x^j} + \frac{\partial u_2^j}{\partial x^i}\right) n^j = 0}$$

(general condition on surface neglecting α — immiscible fluids!)

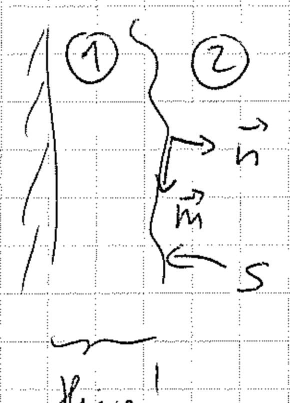
also add mass conservation ...



$$\int_V \frac{D\rho}{Dt}\, dV = \oint_{\partial V = S} \rho\, \vec{u}\cdot d\vec{S}$$

$\Downarrow$

$$\boxed{(\rho_1 \vec{u}_1 - \rho_2 \vec{u}_2)\cdot\vec{n} = 0}$$



- velocity mostly // S ($\perp \vec{n}$)

(equ 4) $\cdot \vec{m}$

$$n^j \mu_1 \left(\partial_j u_1^i + \partial_i u_1^j\right) \cdot m^i = \underbrace{n^j \mu_2 \left(\partial_j u_2^i + \partial_i u_2^j\right) \cdot m^j}_{\substack{\text{if } u_2 \approx 0 \\ \text{or } \mu_2 \ll \mu_1}}$$

$$\mu_1 \left(\nabla_\perp u_1^{\parallel} + \nabla_\parallel u_1^{\perp}\right) =$$

$$= \mu_2 \left(\nabla_\perp u_2^{\parallel} + \nabla_\parallel u_2^{\perp}\right)$$

if u's are mostly //

$$\mu_1 \nabla_\perp u_1^{\parallel} = \mu_2 \nabla_\perp u_2^{\parallel}$$

$$\underbrace{\nabla_\perp u_1^{\parallel}}_{=\,0} = \frac{\mu_2}{\mu_1} \nabla_\perp u_2^{\parallel} \to 0 \quad \frac{\mu_2}{\mu_1} \to 0$$

hence b.c. in hw 3 !!

(equ 4.) $\cdot \vec{n}$ $\longrightarrow$ by same assumption ∂u's drop

$\rightarrow$ $\underline{\underline{p_1 = p_2}}$ $\leftarrow$ another b.c.