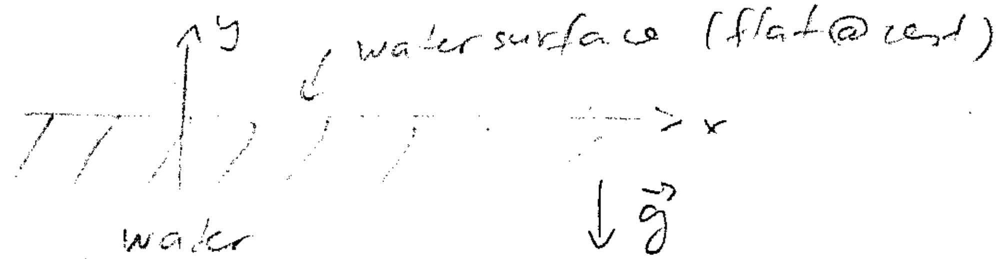
Surface waves ("deep water")

assumptions:
- vorticity is zero (say fluid motion starts @ rest)
- inviscid
- 2d:

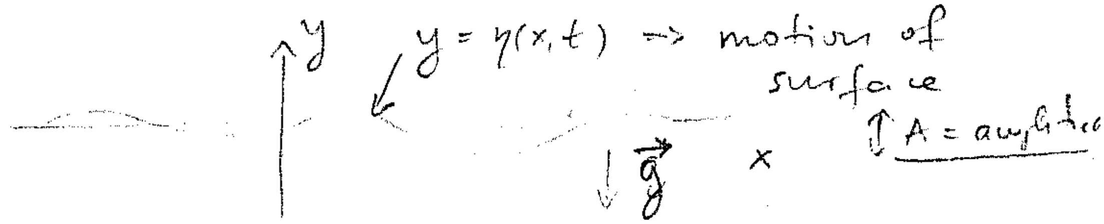


$$\vec{u} = (u_x(x,y,t),\ u_y(x,y,t))$$

$$\exists\ \phi: \quad \vec{u} = \vec{\nabla}\phi \quad (\bar{\nabla}\times\bar{u} = 0)$$

- incompressible

$$\bar{\nabla}\cdot\bar{u} = 0 \Rightarrow \Delta\phi = 0.$$



- a fluid particle on surface has coordinates $(y(t), x(t))$

& moves w/ $\dot{y}(t) = u_y|_\eta$, $\dot{x}(t) = u_y|_\eta$ } particle velocity = local fluid velocity

hence $y(t) = \eta(x(t), t) \quad \forall t \Rightarrow$

(2)

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} + \frac{\partial y}{\partial t}$$

Checks: (a) if $\frac{\partial y}{\partial x} = 0$ free surface horizontal $u_y = \frac{\partial y}{\partial t}$ ✓

(b) if $\frac{\partial y}{\partial t} = 0$: stationary surface slope of surface = slope of streamline

$$\text{(1)} \quad \underline{u_y\Big|_{\eta} = u_x \frac{dy}{dx} + \frac{\partial y}{\partial t}} \iff \text{kinematic condition on surface}$$

• we also found much earlier that N.S.

$$\dot{\vec{u}} + \underbrace{(\vec{u}\cdot\vec{\nabla})\vec{u}} = -\vec{\nabla}\Big(\frac{p}{\rho} + gy\Big)$$

recall $(\bar{\nabla}\times\bar{u})\times\bar{u} = (\bar{u}\cdot\bar{\nabla})\bar{u} - \bar{\nabla}\Big(\frac{\bar{u}^2}{2}\Big)$

so $(\bar{u}\cdot\bar{\nabla})\bar{u} = \underset{\overset{\shortparallel}{0}}{(\bar{\nabla}\times\bar{u})}\times\bar{u} + \bar{\nabla}\Big(\frac{\bar{u}^2}{2}\Big)$

<u>irrotational</u>

body force / unit mass

$\vec{F} = -\vec{\nabla}\psi = -\vec{\nabla}(gy)$

$$\text{so} \quad \dot{\bar{u}} = -\vec{\nabla}\Big(\frac{p}{\rho} + \frac{\vec{u}^2}{2} + gy\Big)$$

$$\Downarrow$$

$$\vec{\nabla}\Big[\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{\vec{u}^2}{2} + gy\Big] = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{\vec{u}^2}{2} + gy = f(t)$$

$f(t)$ can be anything: $\phi(t) \to \phi(t) + \int^t dt' f(t')$ no change to $\vec{u} = \vec{\nabla}\phi$

we need this in order to impose condition on surface that $p = p_0$ (in other words, we take $p_{air} = 0$)

so we want,

$$\frac{\partial \phi}{\partial t}(x, \eta(x,t), t) + \frac{1}{2}\vec{u}^2(x, \eta(x,t), t) + g\eta(x,t) =$$

$$= \underbrace{\left(f(t) - \frac{p_0}{\rho}\right)}_{\text{choose } = 0 \ \left(f = \frac{p_0}{\rho}\right)}$$

$\underline{\text{b.c}}$: $$\left.\frac{\partial \phi}{\partial t}\right|_\eta + \left.\frac{\vec{u}^2}{2}\right|_\eta + g\eta = 0 \quad (2)$$

Now, we'll look for small-amplitude waves...

[w.r.t. what? (later)]

formally: expand in small velocities & amplitudes

we had: (1) $$u_y|_\eta = \underbrace{u_x|_\eta \frac{\partial \eta}{\partial x}}_{\text{higher order}} + \frac{\partial \eta}{\partial t}$$

so $$u_y|_\eta \simeq \frac{\partial \eta}{\partial t}$$

$$u_y(x, \eta(x,t), t) = \frac{\partial \eta}{\partial t}(x,t)$$

$$u_y(x,0,t) + \eta\frac{\partial u_y}{\partial y} + \ldots = \frac{\partial \eta}{\partial t}$$

$$\frac{\partial \eta}{\partial t} \simeq u_y(x,0,t)$$

③ $$\boxed{\frac{\partial \eta}{\partial t} \simeq \frac{\partial \phi}{\partial y}(x,0,t)}$$

(2) gives, neglecting $\vec{u}^2$

$$\left.\frac{\partial\phi}{\partial t}\right|_\eta + g\eta = 0 \Rightarrow \boxed{\frac{\partial\phi}{\partial t}(x,0,t) + g\eta(x,t) = 0}$$

4.

$$\frac{\partial\phi}{\partial t}(x,\underbrace{\eta}_{0},t) + g\eta = 0$$

So we have 3 & 4

$$(**)\ \begin{cases} \dfrac{\partial\eta(x,t)}{\partial t} = \dfrac{\partial\phi(x,0,t)}{\partial y} \\ \dfrac{\partial\phi}{\partial t}(x,0,t) + g\eta(x,t) = 0 \end{cases}$$

$\eta(x,t) = A\cos(\omega t - kx)$ → a harmonic wave on surface.

$(**) = ?$

$$\begin{vmatrix} -A\omega\sin(\omega t - kx) = \dfrac{\partial\phi}{\partial y}(x,0,t) \\ -gA\cos(\omega t - kx) = \dfrac{\partial\phi}{\partial t}(x,0,t) \end{vmatrix}$$

$\Rightarrow \phi(x,y,t) = B(y)\sin(\omega t - kx)$ so $\left.\frac{\partial}{\partial t}\phi\right|_{y=0} \sim \cos$

$+\ (\partial_x^2 + \partial_y^2)\phi = 0$ (because it must) $\quad \left.\frac{\partial}{\partial y}\phi\right|_{y=0} \sim \sin$

$$\Rightarrow \left(-k^2 B(y) + B''(y)\right)\sin(\omega t - kx) = 0$$

$$B'' = k^2 B$$

$$B(y) = \alpha e^{ky} + \beta e^{-ky}$$

$$\phi(x,t) = \left(\alpha e^{ky} + \beta e^{-ky}\right)\sin(\omega t - kx)$$

let depth be @ h : $y = -h$ require $u_y = \frac{\partial \phi}{\partial y} = 0 \Rightarrow$

$$\alpha e^{-kh} - \beta e^{kh} = 0 \Rightarrow$$

$$\Rightarrow \beta = \alpha e^{-2kh}$$

so $\phi(x,t) = \alpha\left(e^{ky} + e^{-2kh-ky}\right)\sin(\omega t - kx)$

(**): $\left.\frac{\partial \phi}{\partial t}\right|_{y=0} = \alpha\omega\left(1 + e^{-2kh}\right)\cos(\omega t - kx) \overset{(**)}{=}$

$$= -gA\cos(\omega t - kx)$$

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = \alpha k\left(1 - e^{-2kh}\right)\sin(\omega t - kx) \overset{(**)}{=} -A\omega\sin(\omega t - kx)$$

so: $\alpha\omega\left(1 + e^{-2kh}\right) = -gA \quad \rightarrow \quad \alpha = -A\frac{g}{\omega}\frac{1}{1+e^{-2kh}}$

$\alpha k\left(1 - e^{-2kh}\right) = -A\omega \quad \quad \underline{\alpha = -A\frac{\omega}{k}\frac{1}{1-e^{-2kh}}}$

hence $\omega^2 = gk\frac{1-e^{-2kh}}{1+e^{-2kh}} = gk\frac{e^{kh}-e^{-kh}}{e^{kh}+e^{-kh}}$

$$\underbrace{\omega^2 = gk\tanh(kh)}$$

$\xrightarrow{\text{deep}}$ $h \to \infty$ $(kh \gg 1)$ $\quad \omega^2 = gk$

$\xrightarrow{\text{shallow}}$ $h \to 0$ $(kh \to 0)$ $\quad \omega^2 = (g^2 h)k^2$

so, deep water waves (of "small" velocity, amplitude..)

$$\omega^2 = gk \qquad \omega = \sqrt{gk}$$

$$c_{phase} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \sim \sqrt{\lambda}$$

- longer waves travel faster !

- $c_{group} = \frac{d\omega}{dk} = \frac{1}{2} c_{phase}$

while shallow water ones are nondispersive

$$\omega = (g^2 h)^{1/2} k$$

$$c = \sqrt{g^2 h}$$

<u>What did "small" mean?</u>

neglected $u_x \frac{\partial y}{\partial x}$ wrt $\frac{\partial y}{\partial t}$ & u_y at $y = \eta$

we have $\phi = \alpha(e^{ky} + e^{-2kh-ky}) \sin(\omega t - kx)$

$$= -A \frac{g}{\omega} \frac{e^{ky} + e^{-2kh-ky}}{1 + e^{-2kh}} \sin(\omega t - kx)$$

and $\eta = A \cos(\omega t - kx)$

$$u_y = \frac{\partial \phi}{\partial y} = -A \frac{g}{\omega} k \frac{e^{ky} - e^{-2kh-ky}}{1 + e^{-2kh}} \sin(\omega t - kx)$$

while
$$u_x = \frac{\partial \phi}{\partial x} \approx A \frac{g}{\omega} k \frac{e^{ky} + e^{-2kh-ky}}{1+e^{-2kh}} \cos(\omega t - kx)$$

so @ $kh \to \infty$ (for simplicity)

$$v_y \sim A k \frac{g}{\omega} e^{ky} \sin(--)$$

$$u_x \sim A k \frac{g}{\omega} e^{ky} \cos(--)$$

$$\frac{\partial \eta}{\partial t} \sim A \omega \cos(--)$$

$$\frac{\partial \eta}{\partial x} \sim A k \sin$$

$$u_x \frac{\partial \eta}{\partial x} \sim A^2 k^2 \frac{g}{\omega} \quad \underline{\text{vs.}} \quad v_y \sim A k \frac{g}{\omega}$$

so $Ak \ll 1$ s.t. l.h.s $\ll$ r.h.s

small amplitude $\Leftrightarrow$ amplitude $\ll$ wavelength

Moral: water waves are fun — dispersive

(even "small amplitude" $A \ll \lambda$)

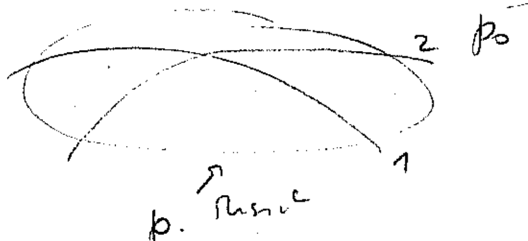
if $kh \gg 1$

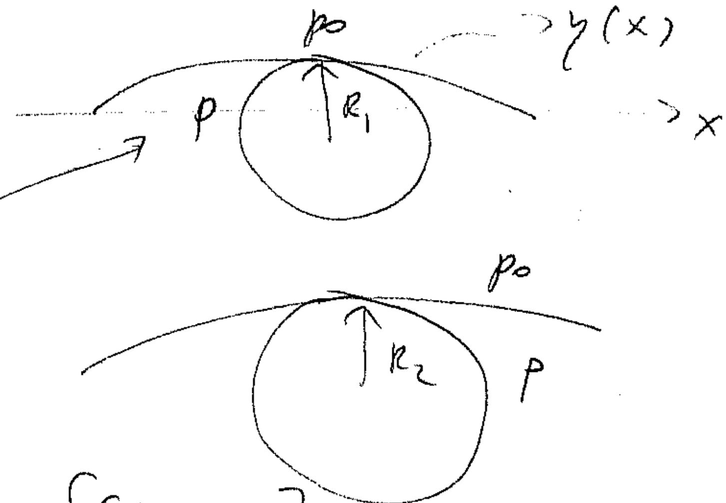
look @ ocean wave patterns --- superpose a few sinusoidal waves & watch!

(>2!)

surface tension ...

$$p - p_0 = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$





- for us $R_2 = \infty$
- $\dfrac{1}{R_1} = -\dfrac{\partial^2 \eta}{\partial x^2}$

on p3 we said $\left.\dfrac{\partial\phi}{\partial t}\right|_\eta + g\eta = \left[f(t) - \dfrac{p_0}{\rho}\right]$ ← take $= 0$ by choice of f

so pressure @ surface should be not p_0 but somewhat higher (for positive $\frac{1}{R_1} = -\frac{\partial^2\eta}{\partial x^2}$)

so we have $$\left.\frac{\partial\phi}{\partial t}\right|_\eta + g\eta = f(t) - \left(\frac{p_0}{\rho} - \frac{\alpha}{\rho}\frac{\partial^2\eta}{\partial x^2}\right)$$

$$\rightarrow \quad \left.\frac{\partial\phi}{\partial t}\right|_\eta + g\eta - \frac{\alpha}{\rho}\frac{\partial^2\eta}{\partial x^2} = 0 \quad \text{(by choice of } f\text{)}.$$

so (**) changed: $$\frac{\partial\phi}{\partial t} = -gA\cos(\omega t - kx) - \frac{\alpha}{\rho}Ak^2\cos(\omega t - kx)$$

$$= A\left(g + \frac{\alpha k^2}{\rho}\right)\cos(\omega t - kx)$$

⇒ deep water only ⇒ $$\omega^2 = gk + \frac{\alpha k^3}{\rho}.$$

cappilary radius 2.7 mm water @ 25 °C.

$$\omega^2 = gk\Big[1 + \underbrace{\frac{\alpha}{g\rho}}_{L^2_{capill}} k^2\Big] = gk\left(1 + L^2_{cap} k^2\right)$$

so if $k = \frac{2\pi}{\lambda}$; $L_{cap} k > 1$

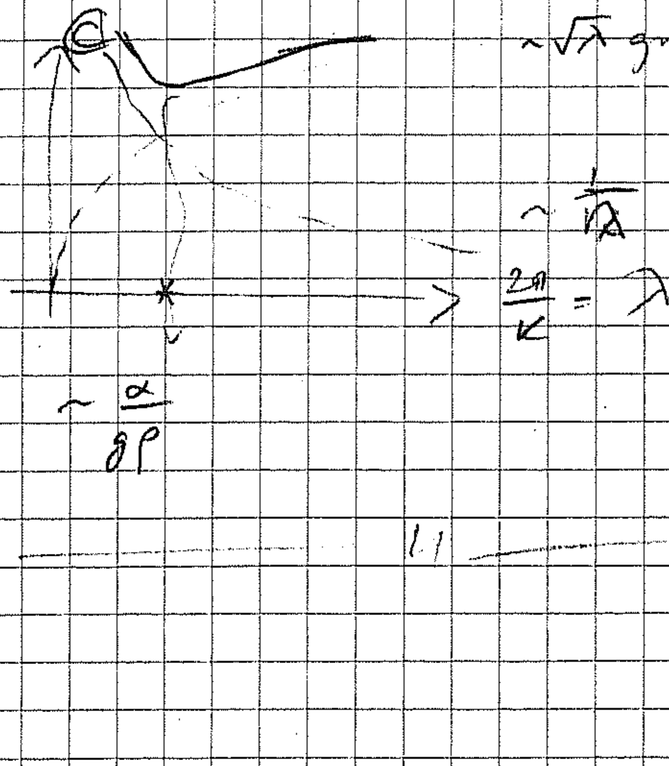
$2\pi L_{cap} > \lambda$

$\lambda \lesssim 2\pi \times 2.7\,mm \sim cm$

surface tension matters.

apparently raindrops & stone falling on a lake produce different waves !!

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}}\sqrt{1 + L_{cap}^2 k^2} \qquad L_{cap} = \frac{\alpha}{g\rho}$$



(1)