More on Zhukovski's map.

$\zeta = f(z)$: analytic map, inverse $z = f^{-1}(\zeta)$

flow: $w(z) \longrightarrow W(\zeta) = w(f^{-1}(\zeta))$

flow in z — flow in ζ — so if $\frac{\partial w}{\partial z} \parallel C$, $\frac{\partial W}{\partial \zeta} \parallel$ is $\frac{f}{C}$

(between small line elements!!)

angles are preserved by $\zeta = f(z)$ map

provided $f'(z) \neq 0$, i.e. at generic points

(they can be doubled, tripled, etc. if

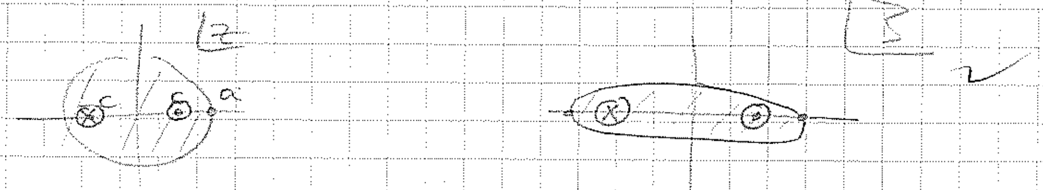
$f' = 0$ but $f'' \neq 0$ or $f' = f'' = 0$, $f''' \neq 0$ etc. ...)

body surface streamline ⇕ map to body surface streamline inverse

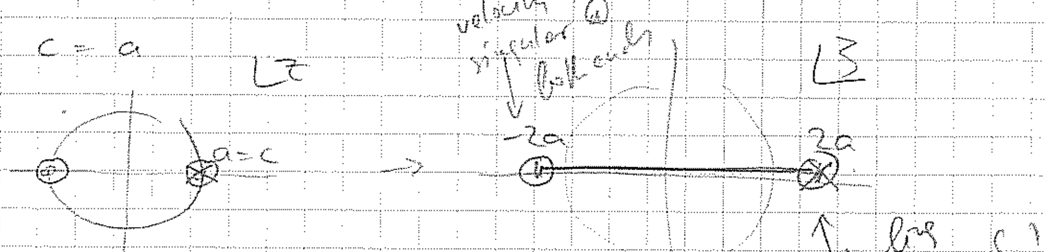
$\zeta = z + \frac{c^2}{z}$ is Zhukovski's map

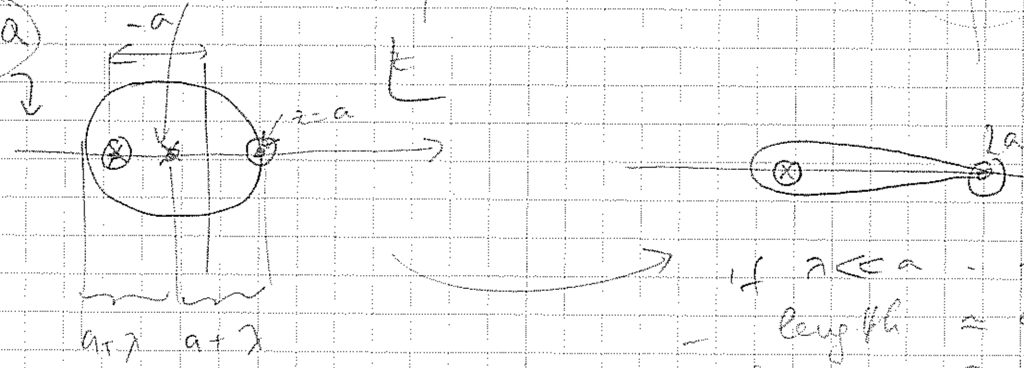
$z = \frac{1}{2}\zeta + \sqrt{\frac{1}{4}\zeta^2 - c^2}$

* a circle w/ radius $a > c$ is mapped to an ellipsoid



* a circle w/ $c = a$





if $\lambda \ll a$. thin aerofoil

– length $\approx 4a$

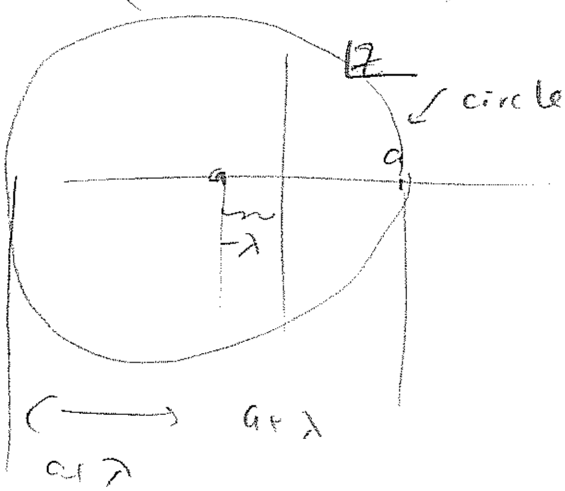
– thick $\sim 3\sqrt{3}\,\lambda$

– $\lambda \ll a$: $\Gamma \approx$ ...

here $\overset{\text{on } C}{z} = -\lambda + (a+\lambda)e^{i\theta} \equiv z|_C$

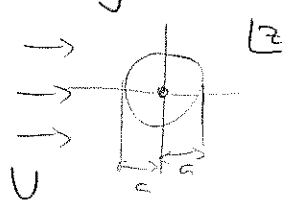
– a circle of radius $a+\lambda$ centered @ $z=-\lambda$

($a > \lambda > 0$)



$$w(z) = \overset{\text{real}}{U}\left(z + \frac{a^2}{z}\right) - i\frac{\Gamma}{2\pi}\log z$$

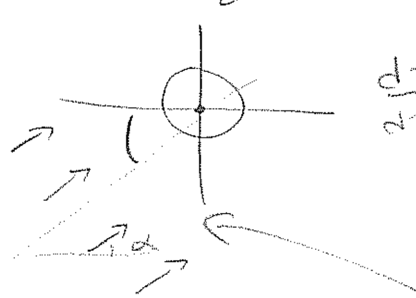
flow w/ circulation ... from



$U \parallel x$

shift center.

$$w(z) = U\left(ze^{-i\alpha} + \frac{a^2}{z}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\log z$$



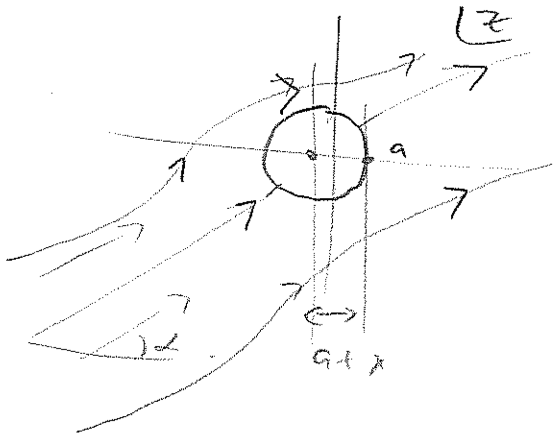
$$\left.\frac{dw}{dz}\right|_{z\to\infty} = Ue^{-i\alpha} = \underbrace{U\cos\alpha}_{u_x} - i\underbrace{U\sin\alpha}_{u_y}$$

$z = ae^{i\theta}$ should be streamline

$$ae^{i(\theta-\alpha)} + ae^{-i(\theta-\alpha)} = 2a\cos(\theta-\alpha) \checkmark$$

so we have same flow but in, i.e. $z \to z+\lambda$

$a \to a+\lambda$



$$w(z) = U\left((z+\lambda)e^{-i\alpha} + \frac{(a+\lambda)^2}{(z+\lambda)}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\log(z+\lambda)$$

flow w/ circulation Γ & U @ ∞ w/ angle α wrt x

Now, Zhukovski map: $\zeta = z + \frac{a^2}{z}$, w/ $a = c$.

where does

$$z\Big|_C = -\lambda + (a+\lambda)e^{i\theta} \quad \text{map to ?}$$

$$\zeta\Big|_C = -\lambda + (a+\lambda)e^{i\theta} + \frac{a^2}{-\lambda + (a+\lambda)e^{i\theta}} =$$

$$= -\lambda + (a+\lambda)e^{i\theta} + \frac{a^2\left(-\lambda + (a+\lambda)e^{-i\theta}\right)}{|(a+\lambda)e^{i\theta} - \lambda|^2} =$$

$$= -\lambda + (a+\lambda)\cos\theta + i(a+\lambda)\sin\theta +$$

$$+ \frac{a^2}{((a+\lambda)\cos\theta - \lambda)^2 + (a+\lambda)^2\sin^2\theta}\left[-\lambda + (a+\lambda)\cos\theta - i\sin\theta(a+\lambda)\right]$$

$\zeta|_C$ for small $\lambda \ll a$

use $$\frac{a^2}{a e^{i\theta} + \lambda(e^{i\theta} - 1)} \approx$$

$$\approx a e^{-i\theta}\left[\frac{1}{1 + \frac{\lambda}{a}(1 - e^{-i\theta})}\right]$$

$$\approx a e^{-i\theta}\left(1 - \frac{\lambda}{a}(1 - e^{-i\theta})\right)$$

$$= a e^{-i\theta} - \lambda e^{-i\theta}(1 - e^{-i\theta})$$

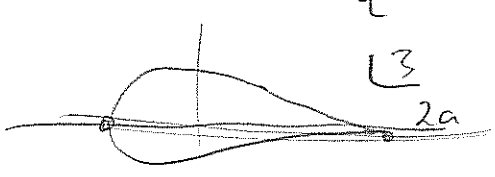
so

$$\zeta|_C \approx -\lambda + (a+\lambda) e^{i\theta} + a e^{-i\theta} - \lambda e^{-i\theta}(1 - e^{-i\theta})$$

$$= -\lambda + (a+\lambda) e^{i\theta} + a e^{-i\theta} - \lambda e^{-i\theta} + \lambda e^{-2i\theta}$$

$$= -\lambda + (a + \lambda + a)\cos\theta - \lambda(\cos\theta - \cos 2\theta)$$
$$+ i[\sin\theta\,(a + \lambda - a + \lambda) - \lambda \sin 2\theta]$$

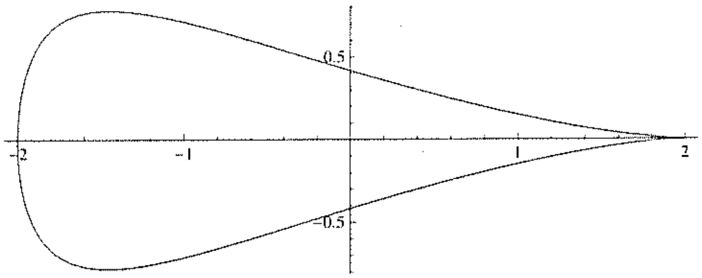
see plot



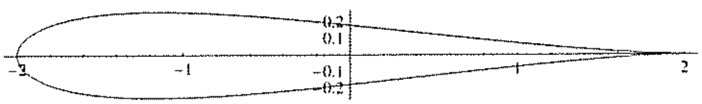
$$\zeta|_{\theta=0} = 2a$$

$$\zeta|_{\theta=\pi} = -\lambda - (2a+\lambda) + 2\lambda \approx -2a - \lambda$$

```
x = - L + (2 + L) Cos[t] - L (Cos[t] - Cos[2 t]);
y = Sin[t] 2 L - L Sin[2 t];
ParametricPlot[{x, y} /. L → .3, {t, 0, 2 Pi}]
```




```
ParametricPlot[{x, y} /. L → .1, {t, 0, 2 Pi}]
```

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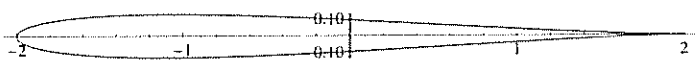
ParametricPlot[{x, y} /. L → .05, {t, 0, 2 Pi}]
```



Now we must map $w(z) \longrightarrow \underline{W}(\zeta)$

it is complicated but we can "cheat" (be smart)

$\underline{W}(\zeta) = w(z(\zeta))$ is $\mathbb{C}$ potential

$$\frac{d\underline{W}}{d\zeta} = \frac{dw(z(\zeta))}{dz}\,\frac{dz(\zeta)}{d\zeta} = \frac{dw}{dz}\,\frac{1}{\frac{d\zeta}{dz}}$$

use $\zeta = z + \frac{a^2}{z}$

$\frac{d\zeta}{dz} = 1 - \frac{a^2}{z^2}$

express as fn of z (easier!)

$$= \left(1 - \frac{a^2}{z^2}\right)^{-1} \times \frac{d}{dz}\left\{U\,(z+\lambda)e^{-i\alpha} + \frac{(a+\lambda)^2}{(z+\lambda)}e^{i\alpha} - \frac{i\Gamma}{2\pi}\log(z+\lambda)\right\}$$

$$= \left(1 - \frac{a^2}{z^2}\right)^{-1}\left(U\left(e^{-i\alpha} - \left(\frac{a+\lambda}{z+\lambda}\right)^2 e^{i\alpha}\right) - \frac{i\Gamma}{2\pi(z+\lambda)}\right).$$

$z = a \Leftrightarrow \zeta = 2a$ singular @ back tip

4.2

Claim: $\exists$ unique value of Γ where no pole @ $z = a$ $\quad (\zeta = 2a)$

basically want numerator to vanish @ $z = a$

$$\frac{dW}{d\zeta} = \frac{z^2}{(z-a)(z+a)} \; \frac{\overbrace{2\pi U e^{-i\alpha}(z+\lambda)^2 - 2\pi U e^{i\alpha}(a+\lambda)^2 - i\Gamma(z+\lambda)}^{(*)\text{ arrange s.t. } =0 \text{ @ } z=a}}{2\pi(z+\lambda)^2}$$

$z=a$ only pole (trailing @ edge)

($z=-a$ inside wing)

($z=-\lambda$ inside wing)

numerator (*)

$$z^2\left[2\pi U e^{-i\alpha}\right] + z\left[-i\Gamma + 4\pi U e^{-i\alpha}\lambda\right]$$

$$+\left[-i\Gamma\lambda - 2\pi U e^{i\alpha}(a+\lambda)^2 + 2\pi U e^{-i\alpha}\lambda^2\right]$$

$$Az^2 + Bz + C = A(z-z_1)(z-z_2)$$

$$\| \quad Az_1z_2 = C$$

$$A(z_1+z_2) = -B$$

if $z_1 = a$, $\quad Aaz_2 = C \Rightarrow z_2 = \dfrac{C}{aA}$

& $A(a+z_2) = -B \Rightarrow z_2 = -\dfrac{B}{A} - a$

$$\Rightarrow \quad \frac{C}{aA} = -\frac{B}{A} - a$$

$$\frac{C}{a} = -B - aA \quad \Rightarrow \quad C = -aB - a^2 A$$

$$-i\Gamma\lambda - 2\pi U e^{i\alpha}(a+\lambda)^2 + 2\pi U e^{-i\alpha}\lambda^2 =$$

$$= i\Gamma a - 4\pi U a e^{-i\alpha}\lambda - 2\pi U a^2 e^{-i\alpha}$$

$$0 = i\Gamma(a+\lambda) - 4\pi U e^{-i\alpha} a\lambda + 2\pi U e^{i\alpha}(a+\lambda)^2$$
$$- 2\pi U e^{-i\alpha}\lambda^2 - 2\pi U e^{-i\alpha} a^2$$

$$-i\Gamma(a+\lambda) = -4\pi U e^{-i\alpha} a\lambda + 2\pi U e^{i\alpha}(a^2 + \lambda^2)$$
$$+ 4\pi U e^{i\alpha} a\lambda - 2\pi U e^{-i\alpha}\lambda^2$$
$$- 2\pi U e^{-i\alpha} a^2$$

$$= 4\pi U a\lambda\, 2i \sin\alpha$$
$$+ 2\pi U a^2\, 2i \sin\alpha$$
$$+ 2\pi U \lambda^2\, 2i \sin\alpha$$
$$= 2i \sin\alpha\, 2\pi U (a+\lambda)^2 .$$

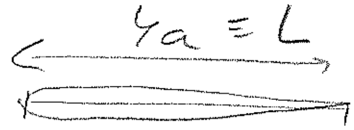
hence

$$\Gamma = -\ 4\pi U(a+\lambda)\sin\alpha .$$

- First, a bit miraculous ---!! (wasn't guaranteed @ all)

Kutte-Zhukovski condition

$\lambda \ll a$ ---



$$\Gamma \approx -4\pi U a \sin\alpha$$

$$\boxed{\Gamma = -\pi U L \sin\alpha}$$

so long as slipping $\overbrace{\text{flow}}^{\text{inviscid}}$ $\overbrace{\text{arrange}}^{\text{can}}$ itself

to smoothly $\to 0$ in a thin layer around!

works well!! (see p 175)