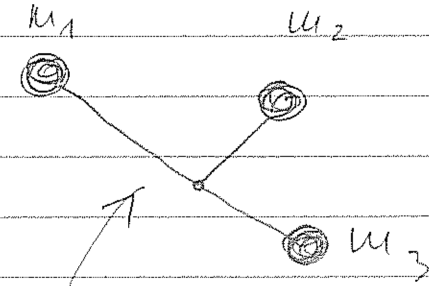
On to Ch. VI – Rigid body motion

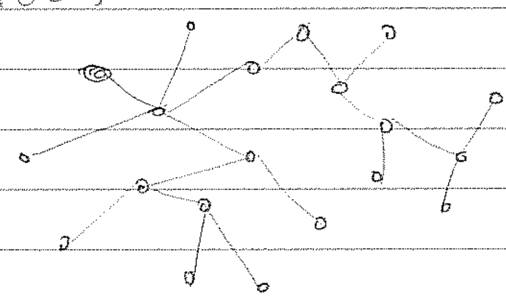
what's a rigid body?

e.g.:



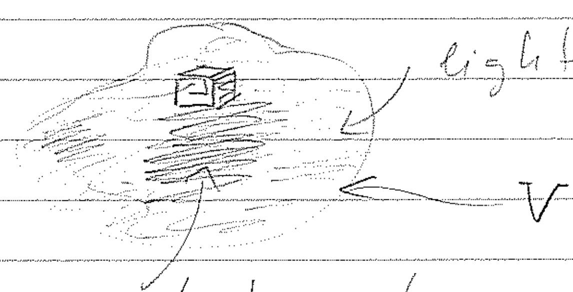
total mass = $m_1 + m_2 + m_3$

or
— many more particles



which don't move relative to each other,

in the case where the dots become a continuum, replace by

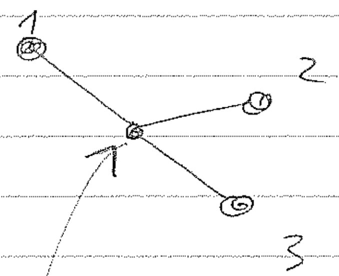


$\rho(\vec{r})\, d^3r$
= # of particles in volume d^3r around $\vec{r}$

total mass $\int_V d^3r\, \rho(\vec{r}) = M$

How can we describe the position of a rigid body?

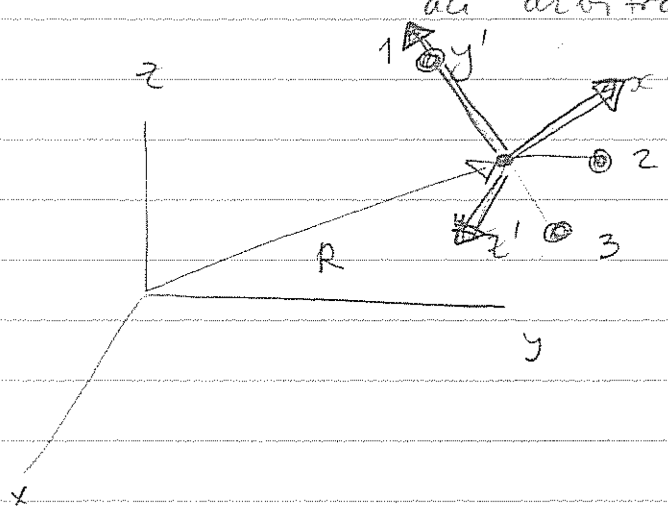
e.g.



(1) some arbitrary point in the body, its coordinate describes the position of the body as a whole —
— could take the c.m., but any other point is OK

— this gives 3 degrees of freedom (i.e. coordinates)

(2) but then, the body can be oriented in an arbitrary way, for example



① $\vec{R}$ chosen point in the body

radius vector of

② (x', y', z') a coordinate system moving with the body (the masses 1, 2, 3 have fixed positions in $x'y'z'$-system)

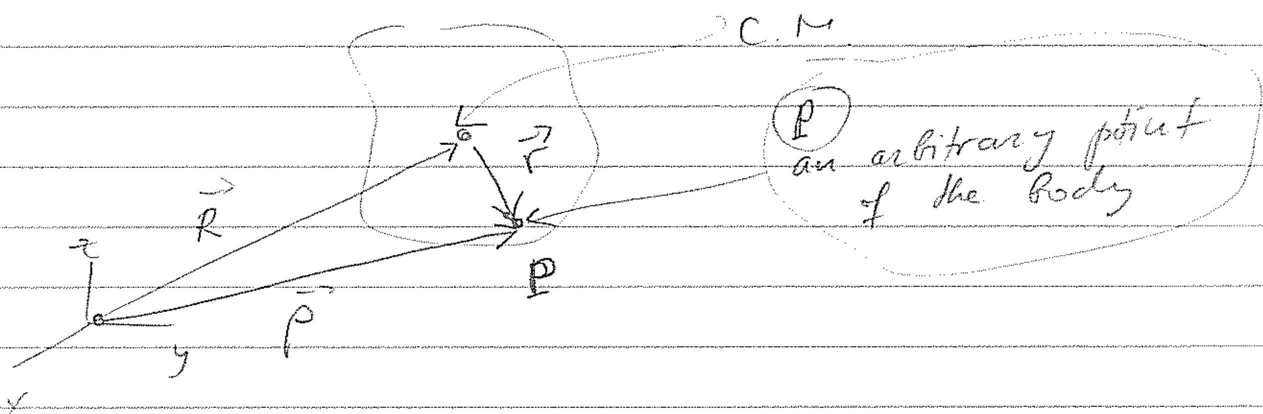
to describe the orientation of $(x'y'z')$ wrt (xyz) system we require 3 other numbers — in general $x'y'z'$ is rotated wrt xyz and a rotation depends on 3 #s so, we need 6 coordinates to describe the motion of the rigid body.

Let us 1st imagine that $\vec{R}$ is the C.M. position

(recall $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$, m_i - the masses comprising the body
$\vec{r}_i$ - their positions in (X, Y, Z) system)

Let an arbitrary point of the body have radius vector

$\vec{\rho}$ in (X, Y, Z) system and $\vec{r}$ in $(x' y' z')$ system



the motion of this arbitrary point consists of
(a) a displacement of C.M. vector $\vec{R}$, $(d\vec{R})$
(b) a rotation of $\vec{r}$ around some direction $\vec{n}$ (axis of rotation) on some angle $\Delta\varphi$ ($d\vec{\varphi} \equiv \vec{n}\, d\varphi$)

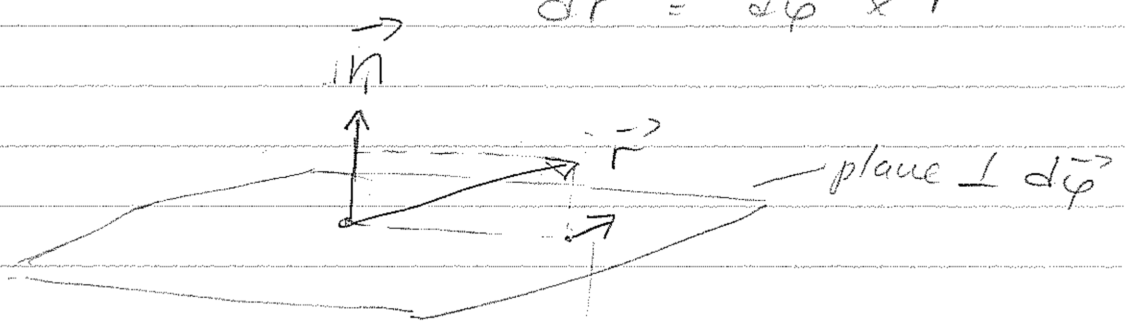
(b) corresponds to a rotation of $x'y'z'$-system wrt xyz system, i.e. a rotation of the entire body around its center of mass thus we have

$$d\vec{\rho} = d\vec{R} + d\vec{r} =$$

$$= d\vec{R} + \underbrace{d\vec{\varphi} \times \vec{r}}$$

remember when we studied rotational invariance & angular momentum we argued

$$d\vec{r} = d\vec{\varphi} \times \vec{r}$$



$$d\vec{\varphi} \times \vec{r} = \vec{n} \times \vec{r}\, d\varphi$$

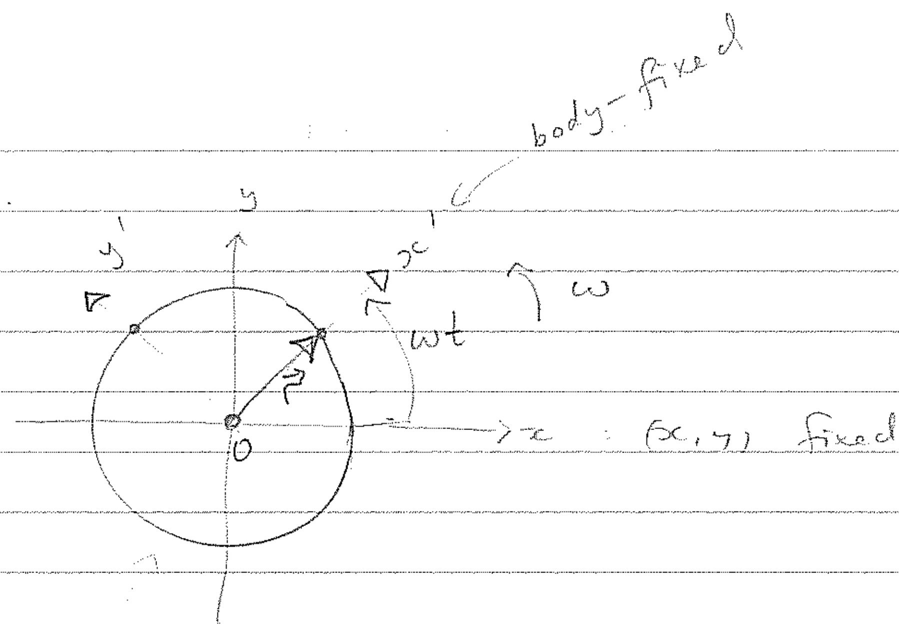
so dividing by dt

$$\underbrace{\frac{d\vec{\rho}}{dt}}_{\substack{\text{velocity} \\ \text{of } \vec{P} \\ \text{in } (x,y,z) \\ \text{frame} \\ \vec{v}}} = \underbrace{\frac{d\vec{R}}{dt}}_{\substack{\text{velocity} \\ \text{of C.M} \\ \vec{V}}} + \underbrace{\frac{d\vec{\varphi}}{dt}}_{\substack{\text{angular} \\ \text{velocity wrt C.M} \\ \vec{\Omega}}} \times \underbrace{\vec{r}}$$

radius vector of $\vec{P}$ wrt (x', y', z') frame (Body rigid - doesn't change)

hence $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$

e.g.



a disk say, rotating w/ ω,

$\vec{r}$: fixed in (x', y') $\quad \vec{r} = (A, 0, 0)$ here, a point a distance A away from origin.

but in (x, y) : $\vec{\rho} = \vec{r} = (A\cos\omega t, A\sin\omega t, 0)$ $\quad (\vec{R}_{CM} = \emptyset)$.

$$\vec{v} = \frac{d\vec{\rho}}{dt} = \omega(-A\sin\omega t, A\cos\omega t, 0)$$

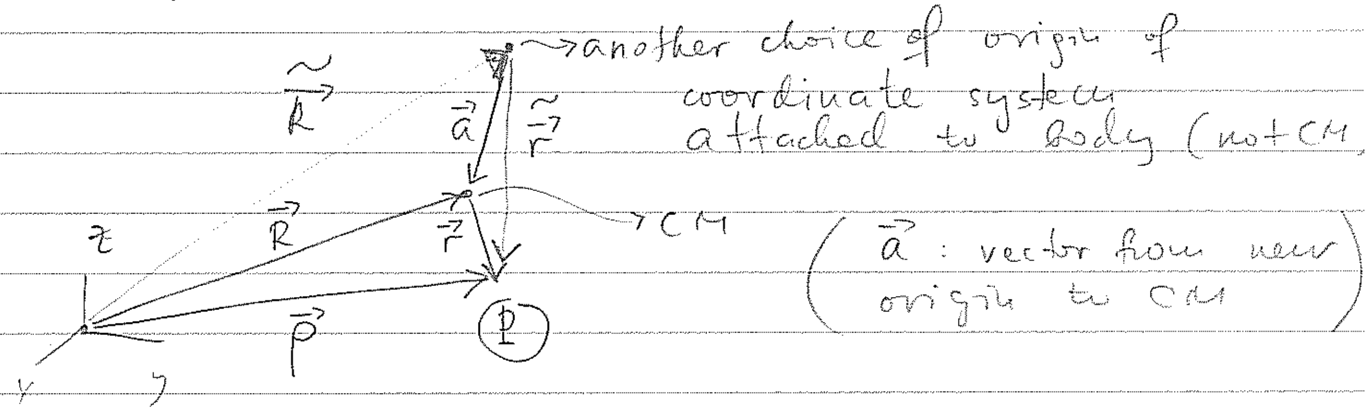
$$= \overbrace{\omega\,(0, 0, 1)}^{\vec{\Omega}} \times \underbrace{(A\cos\omega t, A\sin\omega t, 0)}_{\vec{r}}$$

$$= \vec{\Omega} \times \vec{r}$$

$$\left(\text{since} \quad (\vec{\Omega} \times \vec{r})_x = \overset{0}{\Omega_y} r_z \underbrace{- \Omega_z r_y}_{-\omega A \sin\omega t} \quad \text{etc.} \right)$$

where $\vec{V}$ is the velocity of CM & $\vec{\Omega}$ is the angular velocity of the rotation of the body (here: wrt C.M.).

Now suppose origin is NOT @ C.M. but instead at some point away from C.M.:



we have: $\vec{\rho} = \vec{R} + \vec{r}$

as well as $\vec{\rho} = \tilde{\vec{R}} + \tilde{\vec{r}}$

now we know

$$d\vec{\rho} = d\vec{R} + d\vec{\varphi} \times \vec{r} \quad \text{as already shown}$$

and $\vec{r} = \tilde{\vec{r}} - \vec{a}$ from picture, hence

$$d\vec{\rho} = d\vec{R} + d\vec{\varphi} \times \tilde{\vec{r}} - d\vec{\varphi} \times \vec{a}$$

or $$d\vec{\rho} = \left(d\vec{R} - d\vec{\varphi} \times \vec{a}\right) + d\vec{\varphi} \times \tilde{\vec{r}}$$

divide by dt:

$$\vec{v}_u = \vec{V} - \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \tilde{\vec{r}} \quad (*)$$

Now from $\vec{\rho} = \tilde{\vec{R}} + \tilde{\vec{r}}$ we have

$$(**) \quad \vec{v} = \tilde{\vec{V}} + \tilde{\vec{\Omega}} \times \tilde{\vec{r}}$$

repeat derivative of last equ. on p. 109

where $\tilde{\vec{\Omega}}$ is now the angular velocity of the body in system $(x'y'z')$ w/ origin at $\tilde{\vec{R}}$

compare $(*)$ w/ $(**)$:

$$\tilde{\vec{V}} + \tilde{\vec{\Omega}} \times \tilde{\vec{r}} = \vec{V} - \vec{\Omega} \times \vec{a} + \vec{\Omega} \times \tilde{\vec{r}}$$

since must hold for any $\tilde{\vec{r}}$ (any point in the body)

$$\begin{cases} \tilde{\vec{V}} = \vec{V} - \vec{\Omega} \times \vec{a} \\ \tilde{\vec{\Omega}} = \vec{\Omega} \end{cases}$$

MORAL: if $\tilde{\vec{R}}$ is NOT CM, then

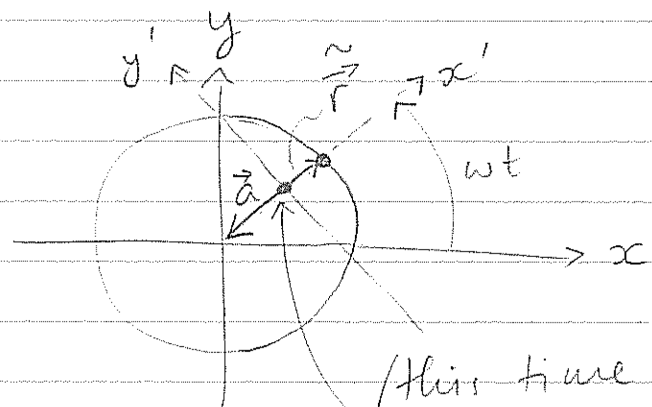
(1) velocity of $\tilde{\vec{R}}$ = C.M. velocity + $\vec{\Omega} \times \underbrace{(-\vec{a})}_{\text{vector from C.M to } \tilde{\vec{R}}}$

(2) <u>angular velocity of frame fixed "in the body" is, at any instant, independent on the chosen origin</u>

→ <u>this velocity is called the "angular velocity of the body"</u>

Ex. ⟹

e.g. in our disk example of p. 109.1



(this time, choose origin of body-fixed system away from c.m.)

now we have $\vec{\rho} = \tilde{\vec{r}} - \vec{a}$

$$\tilde{\vec{r}} = \big((A-a)\cos\omega t, (A-a)\sin\omega t, 0\big)$$

$$\vec{a} = (-a\cos\omega t, -a\sin\omega t, 0)$$

& $\dfrac{d\vec{\rho}}{dt} = -\dfrac{d\vec{a}}{dt} + \dfrac{d\tilde{\vec{r}}}{dt}$

$$= -\vec{\Omega}\times\vec{a} + \vec{\Omega}\times\tilde{\vec{r}}$$

$$= \vec{\Omega}\times(-\vec{a}) + \vec{\Omega}\times\tilde{\vec{r}}$$

$$\vec{v} = \underbrace{\vec{V}}_{\text{velocity of origin of }(x',y')} + \vec{\Omega}\times\tilde{\vec{r}}$$

((N.B.: this is very general (well, we proved it), as at any moment every rotation is a rotation around an axis)) so, now that we can describe kinematics –

– position & velocity of an arbitrary point of the body wrt a fixed coordinate system

– we need to do dynamics... ≡ Lagrangian → E.O.M.

<u>kinetic energy $T = ?$</u> ⇒ Σ of $\frac{mv^2}{2}$ of all particles of the body

∀ particle $\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}$

position of particle wrt origin of $(x'y'z')$

angular velocity of body

velocity of particle in (x,y,z) at a given moment

velocity of origin of $(x'y'z')$ → most common C.M., but NOT always.

i.e. how fast orientation of $(x'y'z')$ changes wrt $(x\,y\,z)$... e.g. $\dot{\varphi}\vec{n}$

this vector $\vec{r}$ has fixed components in (x',y',z') but NOT in (x,y,z)

Imagine body made of individual discrete particles labeled by $\underline{\underline{a}}$

$$\vec{v}_a = \vec{V} + \vec{\Omega} \times \vec{r}_a$$

these are same ∀ particle

let a^{th} particle have mass m_a

$$\underbrace{T_a}_{\text{kin energy } a^{th} \text{ particle}} = \frac{1}{2} m_a \vec{v}_a^{\,2}, \qquad \underbrace{T = \sum_a \frac{1}{2} m_a \vec{v}_a^{\,2}}_{\text{kin. energy of body}}$$

So, for T we have

$$T = \frac{1}{2}\sum_a m_a \vec{v}_a^{\,2} = \frac{1}{2}\sum_a m_a \left(\vec{V} + \vec{\Omega}\times\vec{r}_a\right)^2 =$$

$$= \frac{1}{2}\sum_a m_a\Big(\vec{V}^2 + \underbrace{\vec{V}\cdot(\vec{\Omega}\times\vec{r}_a)}_{=\vec{r}_a\cdot(\vec{V}\times\vec{\Omega})\text{ by cyclic property}} + (\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a)\Big)$$

$$= \frac{1}{2}\vec{V}^2\left(\sum_a m_a\right) + \underbrace{\frac{1}{2}\left(\sum_a m_a \vec{r}_a\right)}\cdot\vec{V}\times\vec{\Omega}$$

this is zero if $(x'y'z')$
has its origin @ C.M. —
— we'll assume this from now on

$$+\frac{1}{2}\sum_a m_a (\vec{\Omega}\times\vec{r}_a)(\vec{\Omega}\times\vec{r}_a) = (*)$$

now $\sum_a m_a = \mu$ = total mass of body

then use $(\bar{A}\times\bar{B})\cdot(\bar{C}\times\bar{D}) = (\bar{A}\cdot\bar{C})(\bar{B}\cdot\bar{D})$
$\qquad\qquad - (\bar{A}\cdot\bar{D})(\bar{B}\cdot\bar{C})$

to rewrite $(\vec{\Omega}\times\vec{r}_a)\cdot(\vec{\Omega}\times\vec{r}_a) = \Omega^2\vec{r}_a^{\,2} - (\vec{\Omega}\cdot\vec{r}_a)^2$

$$(*) = \frac{1}{2}\mu\vec{V}_{C.M.}^{\,2} + \frac{1}{2}\sum_a m_a\left(\vec{\Omega}^2\vec{r}_a^{\,2} - (\vec{\Omega}\cdot\vec{r}_a)^2\right) = (**)$$

last term in $(\ldots)$ $\Rightarrow$ $\vec{\Omega}^2 = \sum_{i=1}^{3}\Omega_i^2 \qquad \vec{\Omega}\cdot\vec{r}_a = \sum_{i=1}^{3}\Omega_i r_{a_i}$

$\vec{r}_a = \{r_{a_i}\} = \underbrace{(r_{a_1}, r_{a_2}, r_{a_3})}$ $\qquad \vec{r}_a^{\,2} = \sum_{i=1}^{3} r_{a_i}^2 \qquad$ &

((instead of using x,y,z → 1,2,3))

$$(\vec{\Omega}\cdot\vec{r}_a)^2 = \left(\sum_{i=1}^{3} \Omega_i r_{a_i}\right)^2 =$$

$$= \left(\sum_{i=1}^{3} \Omega_i r_{a_i}\right)\left(\sum_{j=1}^{3} \Omega_j r_{a_j}\right) =$$

$$= \sum_{i,j=1}^{3} \Omega_i \Omega_j r_{a_i} r_{a_j}$$

this is a "dummy" index — call it anything (k, i) ...

$$\vec{\Omega}^2 \vec{r}_a^{\,2} = \left(\sum_{i=1}^{3} \Omega_i^2\right)\left(\sum_{j=1}^{3} r_{a_j}^2\right) = \sum_{i=1}^{3} \Omega_i \Omega_i \sum_{k=1}^{3} r_{a_k}^2$$

$$\text{so} \quad \vec{\Omega}^2 \vec{r}_a^{\,2} - (\vec{\Omega}\cdot\vec{r}_a)^2 = \sum_{i,j=1}^{3} \delta_{ij}\Omega_i\Omega_j\left(\sum_{k=1} r_{a_k}^2\right) - \Omega_i\Omega_j r_{a_i} r_{a_j}$$

(these two are the same as $\sum_{i,j=1} \delta_{ij} \Omega_i \Omega_j = \sum_{i=1} \Omega_i \Omega_i$

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$)

$$\text{so} \quad \vec{\Omega}^2 \vec{r}_a^{\,2} - (\vec{\Omega}\cdot\vec{r}_a)^2 = \sum_{i,j=1}^{3} \Omega_i \Omega_j \left(\delta_{ij}\sum_{k=1} r_{a_k}^2 - r_{a_i} r_{a_j}\right)$$

finally back to (**) =

$$= T = \frac{1}{2}\mu \vec{V}_{CM}^{\,2} + \frac{1}{2}\sum_a m_a \sum_{i,j=1}^{3} \Omega_i \Omega_j \left(\delta_{ij}\vec{r}_a^{\,2} - r_{a_i} r_{a_j}\right) =$$

$$= \frac{1}{2}\mu \vec{V}_{CM}^{\,2} + \frac{1}{2}\sum_{i,j=1}^{3} \Omega_i \Omega_j \underbrace{\left(\sum_a m_a \left(\delta_{ij}\vec{r}_a^{\,2} - r_{a_i} r_{a_j}\right)\right)}$$

this is a geometric characteristic of the body: depends on masses & their distribution ONLY $\equiv I_{ij}$ ⟶

$$I_{ij} = \sum_a m_a \left(\delta_{ij}\, \vec{r}_a^{\,2} - r_{a_i} r_{a_j}\right)$$

$\sum$ over all particles

i^{th} component ($i = x, y, z$ or $1, 2, 3$) of the coordinate of a-th particle in $(x'y'z')$ system whose origin assumed to be C.M.

inertia **tensor** of the body

for us, it means a simple thing

$$\vec{r} \equiv \text{vector} = \text{column}: \underbrace{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}}_{\equiv r_i} \rightarrow \text{vector} \equiv \text{rank-1 tensor}$$

one index object $\equiv$ $\equiv$ 3 components

I_{ij} = **matrix** (3x3, here)

$\equiv$

2 indices $\equiv$ 2nd rank tensor

In our case $I_{ij} = I_{ji}$ —

(obvious — e.g. $I_{12} = -\sum_a m_a r_{a_1} r_{a_2}$

$= -\sum_a m_a r_{a_2} r_{a_1}$

$= I_{21}$ etc. . .)

– symmetric tensor of 2nd rank
(symmetric matrix, 3x3)

Now, from linear algebra, you know that if you're given a symmetric $n \times n$ matrix, it can always be diagonalized by an orthogonal $(n \times n)$ transform ...

-- here: $n = 3$

3×3 orthogonal matrix

$$I^{diag.} = O\, I\, O^T$$

diagonal 3×3 m-x

in matrix notation: 3×3 symmetric matrix w/ m-x elements I_{ij}

i.e. inertia tensor can be diagonalized by an orthogonal transformation

the elements of $I^{diag} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

are called "principal moments of inertia"

$\equiv$ eigenvalues of the 3×3 symmetric $\|I_{ij}\| = I$

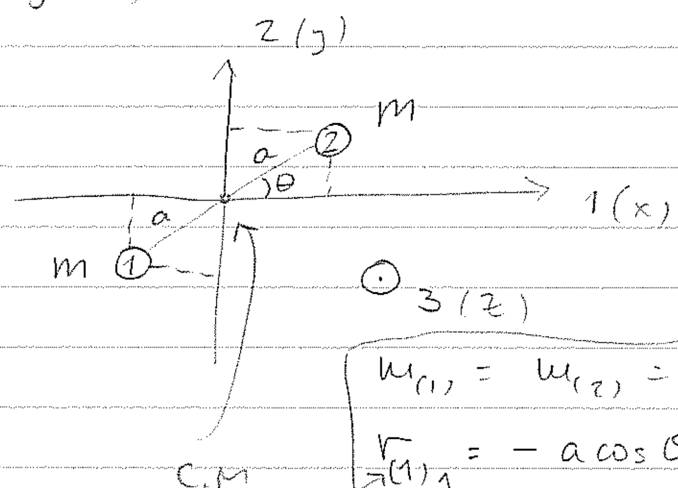
the orthogonal transformation O corresponds to a change of orientation of $(x'y'z')$ which ensures that in the rotated system I is diagonal.

The corresponding directions are called "principal axes of inertia"

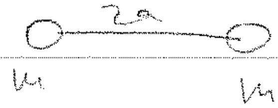
(i.e. $\hat{x}'\hat{y}'\hat{z}'$ of system where I is diagonal)

Usually, it is evident from the symmetries of the body what the principal axes of inertia are — but for a body of an arbitrary shape one would have to rely on the math.

Here's an example of an "unfortunate" choice of $(x'y'z')$:



diatomic molecule considered as rigid body



$$m_{(1)} = m_{(2)} = m$$

$$r_{(1)_1} = -a\cos\theta \qquad r_{(2)_1} = a\cos\theta$$

$$r_{(1)_2} = -a\sin\theta \qquad r_{(2)_2} = a\sin\theta$$

$$r_{(1)_3} = 0 \qquad r_{(2)_3} = 0$$

$$\vec{r}_{(1)}^{\,2} = a^2 = r_{(2)}^2$$

a=(1) or (2)

a=2

$$I_{ij} = \sum_a m_a\left(\vec{r}_a^{\,2}\,\delta_{ij} - r_{a_i} r_{a_j}\right)$$

so in this system we have

$$I_{11} = \overset{①}{m(a^2 - a^2\cos^2\theta)} + \overset{②}{m(a^2 - a^2\cos^2\theta)}$$

$$= 2ma^2\sin^2\theta$$

$$I_{12} = I_{21} = -ma^2\cos\theta\sin\theta$$

$$I_{22} = m(a^2 - a^2\sin^2\theta) + m(a^2 - a^2\sin^2\theta)$$

$$= 2ma^2\cos^2\theta$$

$$I_{13} = I_{31} = 0 \;,\; I_{23} = I_{32} = 0 \quad (\text{since } r_3^{(1,2)} \equiv 0)$$

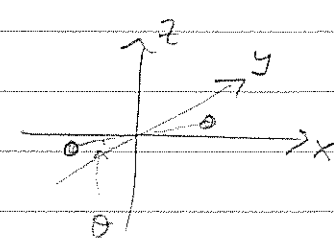
$$I_{33} = 2ma^2$$

So

$$\underline{I} = 2ma^2 \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A

<u>math</u>: Eigenvalues of ( )



are 1 & λ: $\underbrace{(\sin^2\theta - \lambda)(\cos^2\theta - \lambda) - \sin^2\theta\cos^2\theta = 0}$

$$\lambda^2 - \lambda(\sin^2\theta + \cos^2\theta) = 0$$

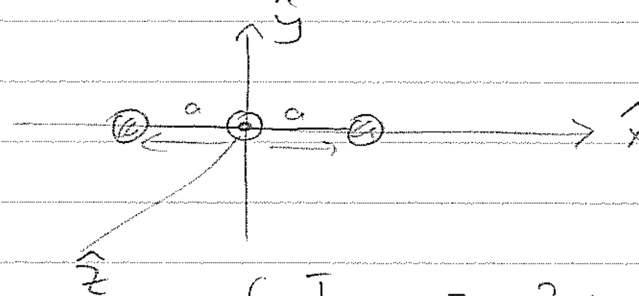
$$\lambda = 0$$
$$\lambda = \underline{1}$$

find θ using lin. algebra

hence there exists a coordinate frame where

$$\underline{I}' = 2ma^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(B) physics — silly choice of orientation



→ or some from $\theta = 0$ value of I from p. 118

from 1st year physics
$$\begin{cases} I_{33} = 2ma^2 \\ I_{11} = 0 \\ I_{22} = 2ma^2 \end{cases}$$

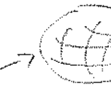
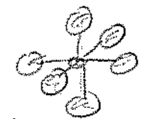
$$I = 2ma^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So generally we have

$$T = \frac{1}{2}\mu \vec{V}_{CM}^{\,2} + \frac{1}{2}\sum_{i=1}^{3} \underbrace{\Omega_i \Omega_j}_{\text{components of angular velocity of body}} \overbrace{I_{ij}}^{\text{components of inertia tensor}}$$

or in a $(x' y' z')$ where axes are aligned w/ the principal axe of inertia:

$$T = \frac{1}{2}\mu \vec{V}_{CM}^{\,2} + \frac{1}{2}\underbrace{\left(\Omega_1^2 I_1 + \Omega_2^2 I_2 + \Omega_3^2 I_3\right)}_{\text{generally } I_1\, I_2\, I_3 \text{ are different}}$$

 or 

– they are all the same for a sphere, for example

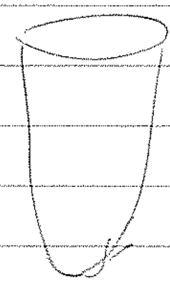
(we call such a body a "spherical top")

– if two are equal – like for our ,

we have a "symmetrical top"

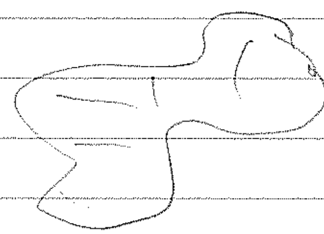
this one, where $I_1 = 0$
$\equiv$ "rotator"

such bodies have cylindrical symmetry

 or 

– and the more general case is an

"asymmetrical top"



finally:

– "parallel axes theorem": I_{ij} in C.M. $\xrightarrow{?}$

$\to I'_{ij}$ in $\vec{r}_a = \vec{r}_a' + \vec{b}$

constant vector – shift of origin by $\vec{b}$

$I'_{ij} = \sum_a m_a \left(\delta_{ij}\, \vec{r}_a'^{\,2} - r'_{a_i} r'_{a_j}\right) = \longrightarrow$

$$= \sum_a m_a \Big(\delta_{ij} \big(\vec{r}_a^{\,2} - \underbrace{2 \vec{r}_a \cdot \vec{b}}_{\sim \vec{b}\cdot\sum_a m_a \vec{r}_a = 0} + \vec{b}^{\,2} \big) - r_{a_i} r_{a_j} - r_{a_i} b_j - r_{a_j} b_i - b_i b_j \Big) =$$

since $\vec{r}_a$ are C.M.

vanish for same reason $\sum_a m_a r_{a_i} = 0$

$$= \sum_a m_a \left(\delta_{ij} \vec{r}_a^{\,2} - r_{a_i} r_{a_j} \right) +$$

$$+ \sum_a m_a \left(\vec{b}^{\,2} \delta_{ij} - b_i b_j \right) = \underbrace{I_{ij}}_{CM} + \mu \, \delta_{ij} \left(\vec{b}^{\,2} - b_i b_j \right)$$

$$= I'_{ij}$$

$$\left(\text{shifted} : \vec{r}_a = \vec{r}_a^{\,\prime} + \vec{b} \right)$$

$\boxtimes$

→ What is L w/ external potential?

-- well: $\underline{L = T - U}$

$$T = \frac{1}{2} \mu \vec{V}_{CM}^{\,2} + \frac{1}{2} \left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \right)$$

U depends on $\underbrace{\vec{R}_{CM}}_{\text{e.g. representing force applied to body as a whole}}$ as well as $\underbrace{\text{orientation of } (x'y'z') \text{ wrt } (xyz)}_{\text{as we'll see, will represent torques acting on body}}$

For bodies w/ continuous mass density $\rho(\vec{r})$

$$\underbrace{\rho(\vec{r})\, d^3\vec{r}}_{\text{mass inside}} \qquad dx\,dy\,dz = d^3\vec{r}$$

$$I_{ij} = \int_V \rho(\vec{r}) \left(\delta_{ij} \vec{r}^{\,2} - r_i r_j\right) d^3\vec{r} \qquad (\ast\ast\ast)$$

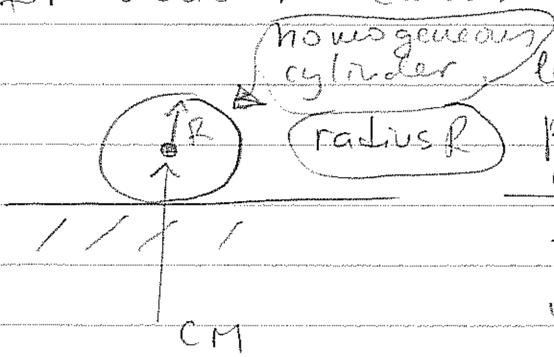
volume of body

(which is simply the continuum version of

$$I_{ij} = \sum_a m_a \left(\delta_{ij} \vec{r}_a^{\,2} - r_{a_i} r_{a_j}\right)$$
)

& T is given by same expression, w/ corresponding I_{ij} from $(\ast\ast\ast)$ & $\mu = \int_V \rho(\vec{r})\, d^3\vec{r}$

Our formulae so far can be used to calculate T for various cases -- e.g.



let's say moment of inertia I around principal axes // axes of cylinder and mass is μ.

let's say it's rolling on a plane w/out slipping