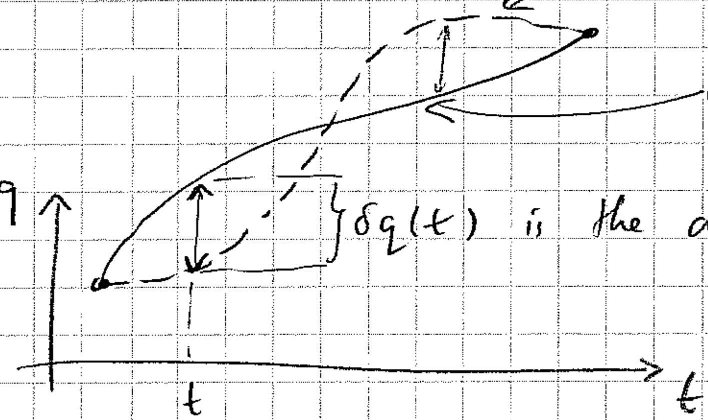
A few comments sparked by questions:

① another way to think of $\delta q(t)$ as "small" is as follows



$\}\delta q(t)$ is the difference between q' & q $\forall t$

to make sure it's "small" write $q'(t) = q(t) + \varepsilon\,\delta q(t)$ & take $\varepsilon \to 0$.

Then

$$\delta S[q] = S[q + \varepsilon\,\delta q] - S[q] =$$

$$= \varepsilon \int dt\, \delta q\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right) + \mathcal{O}(\varepsilon^2)$$

This shows this term is linear in ε ≡

≡ what we mean when we say "linear in δq"

(their def. of δS)

[Also, note L & L mean that δS = linear term only = 0]

(2) 13.2

About the "main theorem of variational calculus"

says that

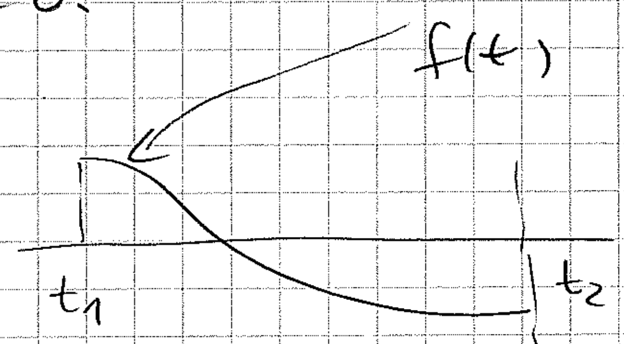
if we have $\displaystyle\int_{t_1}^{t_2} f(t)\, g(t)\, dt = 0$

$f(t)$: some $f(t)$

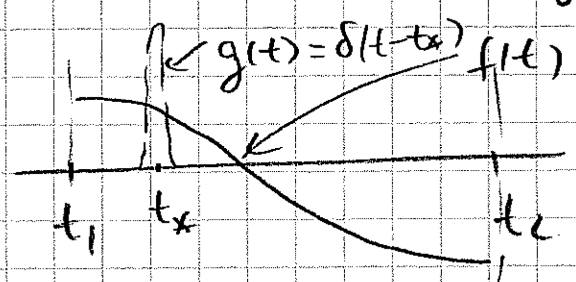
$g(t)$: arbitrary, $g(t_1) = g(t_2) = 0$

$\Rightarrow \quad f(t) \equiv 0.$

Intuitively, take



but then $\int f \cdot g = 0 \quad \forall g \rightarrow$ take $g(t)$ to be δ-function like



$$0 = \int g(t) f(t)\, dt = \int \delta(t - t_*) f(t)\, dt = f(t_*) \quad - \quad \forall t_*$$

$$\Rightarrow \quad f(t) = 0$$

Ex: for $L = T - U = \frac{m\dot{q}^2}{2} - U(q)$

$$\frac{\partial L}{\partial q} = -U'(q)$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \text{E.-L. eqn}$$

$$\Updownarrow$$

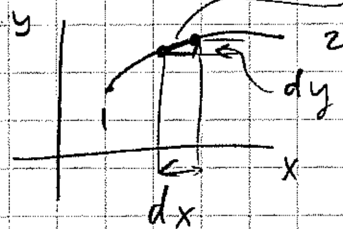
$$\underbrace{-U'(q) = \frac{d}{dt}\, m\dot{q} = m\ddot{q}}_{\text{Newton's equation}}$$

What we've shown is that Newton's equations — differential equ's of second order, allowing us to use initial position & velocity to find position & velocity at any later (earlier) time — are equivalent to a different formulation, where trajectory is determined by minimizing the action functional.

How does the particle "know" to minimize action? (as it must "know" where it'll end up!?)

(action depends on initial & final points)

Another ex. of variation's use (14.1)



$ds = \sqrt{dx^2 + dy^2}$

$$l(1,2) = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

what's shortest curve $1 \to 2$? $\Rightarrow$ min $l(1,2)$

$l(1,2)$ like action

x like time (t)

$y(x)$ like $q(t)$ trajectory

$\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ like $L(\cancel{q}, \dot{q}, \cancel{t})$

only

so $\frac{\partial L}{\partial q} = 0$ $\quad \frac{\partial L}{\partial \dot{q}} \Rightarrow \frac{d}{d\left(\frac{dy}{dx}\right)} \sqrt{1+\left(\frac{dy}{dx}\right)^2} =$

$$= \frac{dy/dx}{\sqrt{1 + dy/dx}}$$

E.-L.

$$0 = \frac{d}{dx} \frac{dy/dx}{\sqrt{1+\frac{dy}{dx}}} \Rightarrow \frac{\frac{dy}{dx}}{\sqrt{1+\frac{dy}{dx}}} = C = \text{const}$$

can be iff $dy/dx = a$, $a = \text{const}$, $\frac{a}{\sqrt{1+a}} = C$

So $\frac{dy}{dx} = a \Rightarrow \underline{y = ax + b}$

a, b determined

by $y(x_1) = y_1$

$y(x_2) = y_2$

(two constants – two eqns).

In classical mechanics, this can not be understood —
— even though the mathematical equivalence between the Euler-Lagrange equations (conditions to extremize action) and Newton's equations of motion can be derived, as we did.

The least action principle can be fully appreciated only if we remember that classical mechanics is only a limit of quantum mechanics. This limit is (formally) obtained by taking $\hbar \to 0$ ($\hbar = \frac{h}{2\pi}$).

But $\hbar$ has <u>dimensions of action</u> (!)

$$\hbar \approx 10^{-34}\ \mathrm{J \cdot s} = 10^{-34}\ \frac{\mathrm{m^2\,kg}}{\mathrm{s}}$$
$$\approx 6.6\ 10^{-16}\ \mathrm{eV \cdot s}$$

So it doesn't make sense to say $\hbar$ is "small" ---
--- really one must compare the action for a given process to $\hbar$, and the classical limit is when $\frac{S}{\hbar} \to \infty$.

Examples ⟶

H atom $\longrightarrow$ speed of electron in ground state

$$V_{el} \approx \alpha c \quad , \quad \alpha = \frac{1}{137} = \frac{e^2}{4\pi\epsilon_0 \hbar c}$$

"fine structure constant" $\Longleftarrow$ dim-less

$$c \approx 3 \times 10^8 \ \frac{m}{s}$$

Bohr radius:

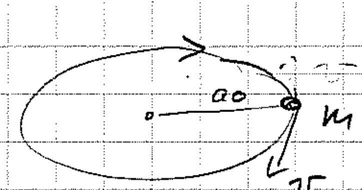
$$a_0 = \frac{\hbar}{m_e c \alpha} \quad \left(\approx \frac{1}{2} \times 10^{-9} \ m \right)$$

$$m_e \approx 10^{-30} \ kg \quad \left(\Rightarrow m_e c^2 \approx 511 \ keV \right)$$

$$\begin{pmatrix}\text{action for} \\ \text{period}\end{pmatrix} \sim \left(\text{kinetic energy} \times \text{period}\right) \sim \left(\int_0^T \frac{mv^2}{2} dt \right)$$

$$\sim \frac{m_e v^2}{2} T \sim \frac{m_e v^2}{2} \frac{2\pi a_0}{v} \approx 2\pi \, m_e v a_0 \frac{1}{2}$$

$\rightarrow$ but $m_e V a_0$ = angular momentum $\quad (\vec{L} = \vec{r} \times \vec{p})$



and we know from QM

that angular momentum is quantized (Bohr-Sommerfeld quantization), more precisely, $m_e v a_0 = \hbar \, 1$

So (action for period) $\approx \pi \hbar$, certainly not $\frac{S}{\hbar} \to \infty$!

Earth/Sun $\longrightarrow$ same formula for action

but $m_e \to M_\otimes$

$v \to v_\otimes$

$T \to 1$ year

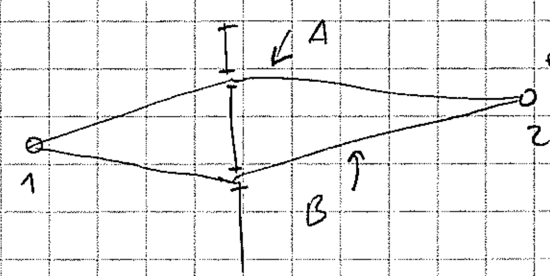
clearly $\underbrace{S_\otimes/\hbar \gg 1}_{\text{(DIY to find \# !)}}$ } clearly classical

— " — " — " —

<u>What does this have to do with the least action principle?</u>

— besides having learned something about why "action" is of interest beyond classical mechanics

In QM, a particle "<u>goes through all paths</u>"



e.g. double-slit experiment —

— at 2, constructive interference if "in phase" or destructive interference if "out of phase"

Feynman formulated QM in terms of a "sum over paths" — Feynman path integral

— and showed that the amplitude for a particle to go from 1 to 2 is proportional to

$$\sum_{\substack{\text{all}\\ \text{paths}\\ 1\to 2}} e^{i\frac{S_{path}}{\hbar}}$$

small amplitude – small probability

In the classical limit $\hbar \to 0$ $\left(\frac{S_{path}}{\hbar} \to \infty\right)$ the phase factor is wildly oscillating, meaning that for a generic path a small deviation from it causes $e^{iS/\hbar}$ to change a lot (destructive interference) only for a path which extremizes S the change of $S/\hbar$ will be small (as $\delta S \approx 0$, as we discussed) and one gets constructive interference.

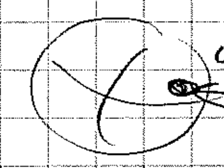
large amplitude → large probability

→ hence the least action path dominates in the classical limit that $S/\hbar \gg 1$

((of course initial state of particle has to be ≈ classical, i.e. a wave packet w/ $\Delta x \Delta p \sim \hbar$, rather than $\Delta x \Delta p \gg \hbar$ ---))

To finish off this diversion away from our main topic, let's mention other reasons why the Euler-Lagrange (least action) formulation is superior to Newton's equs & why it is also useful elsewhere:

* in mechanics, least action principle allows for a straightforward formulation of many problems in a way independent of coordinates used (Cartesian, spherical, elliptical...), as well as on curved spaces, e.g. for systems with constraints



* essential in QM, as discussed, e.g. for classical limit
* General Relativity also formulated via least action
* Classical electrodynamics + particles
* Action & Lagrangian main tool in elementary particle physics (Quantum field theory) & string theory

useful tool — —

Now, we've shown that taking

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) \quad \text{\& demanding } \delta S = 0$$

yields E.-L. equs:

$$\frac{\partial L(q,\dot{q},t)}{\partial q} = \frac{d}{dt} \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}}$$

for a single-variable system (one q & $\dot{q}$).

Everything remains the same if we have many q_i's, $i = 1 \ldots 3N$, as for a system of N particles — we just have to vary $q_1, q_2, \ldots, q_{3N}$ independently and obtain $3N$ equations:

$$\underbrace{\frac{\partial L(q_1 \ldots q_{3N}, \dot{q}_1 \ldots \dot{q}_{3N}, t)}{\partial q_i}}_{} = \frac{d}{dt} \frac{\partial L(q_1 \ldots q_{3N} \dot{q}_1 \ldots \dot{q}_{3N}\, t)}{\partial \dot{q}_i}$$

these are the E-L equations for a system of $3N$ particles $\Leftrightarrow$ L is a function of $3N$ q's & $3N$ $\dot{q}$'s & maybe t.

Now, how do we find $\mathcal{L}(q, \dot{q}, t)$?

will skip writing $q_1 - q_{3N}$, will be understood

For one particle in 1d, we argued that

$$L(q,\dot{q}) = \frac{m\dot{q}^2}{2} - U(q)$$

for a particle of mass m moving in a potential $U(q)$, based on equivalence with the Newton's equation for such a particle.

— we tacitly assumed that if $U(q)=0$, i.e. no force ($f=-U'$) acts on the particle, we'd have $L(q,\dot{q}) = \frac{m\dot{q}^2}{2}$ ⇒ i.e. we used an "inertial frame"

In classical mechanics we postulate the existence of inertial frames - frames of reference in which space is homogeneous & isotropic & time is homogeneous.

This means that in an inertial frame, if no forces are acting on a body, it should move on a straight line w/ constant speed

the Galilean principle of relativity postulates that

(1) the laws of nature at all times are the same in all inertial frames

(2) all frames of reference moving w/ constant velocity w.r.t. an inertial frame are inertial (so there's an ∞ number of inertial frames).

((Now - how can we tell when no forces are acting on a body? — only if it is so far away from other bodies that forces are small enough (≈ 0)..

In many cases, a system of reference based on the surface of the Earth is $\approx$ inertial (when one can neglect the gravitational pull of the Earth & the fact that it rotates around its axis & the Sun) — e.g. a bowling ball moving on the surface (2d) would move "indefinitely" w/ constant $\vec{v}$ if not for friction & the curvature of the Earth. In other cases, one has to worry about Earth's motion & base the coordinate system on the Sun — in even more general cases worry about the Sun motion & base it on the stars ...

Luckily, the forces from a given body on another decrease like $\frac{1}{r^2}$ $\left(= G_N \frac{M_1 M_2}{r^2} \; ; \; \overset{1}{O} \overset{r}{\longleftrightarrow} \overset{2}{O} \right)$ hence in many cases small to worry about.))

let's see now how homogeneity & isotropy + Galileo's relativity principle will help us find L for a free particle:

note - while this is a bit abstract, and it'd be OK to just work w/ $L = T - U$, with T & U as we know them, it illustrates an important point useful throughout physics: that symmetry principles determine the dynamics!

Thus, for a free particle in an inertial frame of reference L can not explicitly depend on $\vec{r}$ or t

(recall $L(\dot{\vec{r}}, \vec{r}, t)$ in general) $\longrightarrow L(\dot{\vec{r}})$

homogeneity

for if it did, the E.-L. equations would explicitly depend on $\vec{r}$ & t, thus dynamics @ different t & $\vec{r}$ would be different, contradicting principle of relativity.

isotropy? $\rightarrow$ means $L(\dot{\vec{r}}) = L(\dot{\vec{r}}^{\,2})$ only

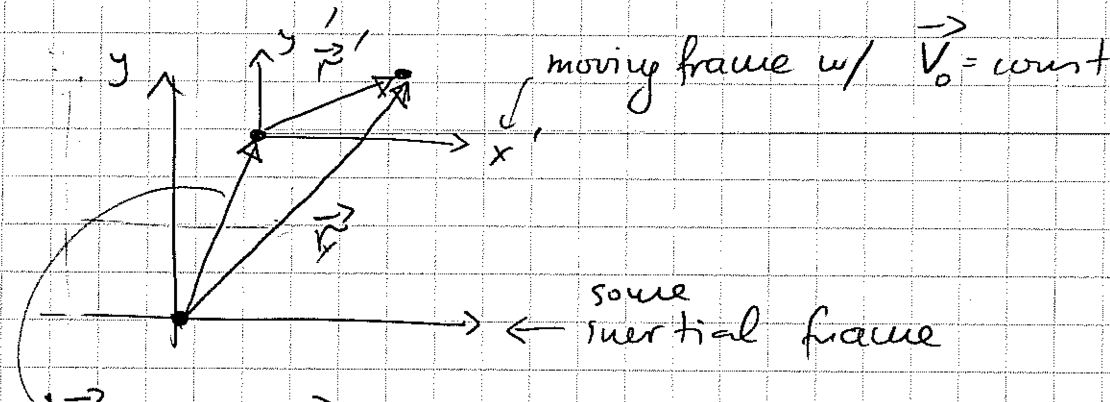
e.g. could NOT have $\underbrace{V_x^2 + \frac{1}{2}V_y^2 + \frac{1}{4}V_z^2}$ but only $V_x^2 + V_y^2 + V_z^2$

this would mean that x, y, z directions differ! $\rightarrow$ NON ISOTROPIC

So from homogeneity & isotropy alone we have

$$L(\bar{r}, \dot{\bar{r}}, t) \longrightarrow L(\dot{\vec{r}}^{\,2})$$

Finally apply Galileo's relativity principle –
– laws of physics ≡ equs. of motion (E.-L.)
should be the same in



$$\vec{R} = \vec{r}_0 + \vec{V}_0 t$$

hence $\quad \vec{r}' = \vec{r} - \vec{R} = \vec{r} - \vec{r}_0 - \vec{V}_0 t$

and $\left(\frac{d}{dt}\right)$: $\quad \vec{v}' = \vec{v} - \vec{V}_0$

and of course $\quad t' = t \quad$ (this is not special relativity
time's the same in all frames
$|V_0| \ll c$)

→ "Galilean transformations" ≡ ($|V_0| \ll c$ limit of Lorentz transformation)

Galilean principle of relativity ⟺ in (x', y', z') one should have E.O.M. i.t.o. $(\bar{v}', \bar{r}')$ which have same form as those in (x, y, z) i.t.o. $(\bar{v}, \bar{r})$

this means that equations of motion following from $L(\vec{v}^2)$ & $L(\vec{v}'^2)$ w/ $\vec{v}' = \vec{v} - \vec{v}_0$

system (x,y,z) coordinates $(\vec{r}, \vec{v})$ $\left\{ \underbrace{\frac{\partial L}{\partial \vec{r}}}_{=0} - \frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = 0 \right.$ $\left\{ \underbrace{\underbrace{\frac{\partial L}{\partial \vec{r}'}}_{=0} - \frac{d}{dt}\frac{\partial L}{\partial \vec{v}'}}_{\text{system } (x'y'z') \text{: coordinates } \vec{r}', \vec{v}'} = 0 \right.$

should be identical.

Note: $$\frac{\partial L}{\partial \vec{v}} \equiv \left(\frac{\partial L}{\partial v_x}, \frac{\partial L}{\partial v_y}, \frac{\partial L}{\partial v_z} \right)$$

$$\frac{\partial L}{\partial \vec{r}} = \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial z} \right)$$

<u>Claim</u>: this is only true if $L(\vec{v}^2) = \text{const}\cdot \vec{v}^2$

(e.g. Galileo's principle holds)

<u>Proof in a few steps:</u>

(1) consider $L(q,\dot{q},t)$ & $L'(q,\dot{q},t) = L(q,\dot{q},t) + \underbrace{\frac{d}{dt} f(q,t)}_{\text{total derivative}}$

then E.O.M. from L & L' – same.

(i.e. two lagrangians differing by total derivative – same E–L equs.)

this follows from $S = \int_{t_1}^{t_2} dt\, L(q\dot{q}t)$

$$S' = \int_{t_1}^{t_2} dt\, L'(q\dot{q}t) = \int_{t_1}^{t_2} dt\, L(q\dot{q}t) + \int_{t_1}^{t_2} dt \frac{d}{dt} f(q,t) =$$

$$\Rightarrow \quad S' = S + \underbrace{f(q(t_2),t_2) - f(q(t_1),t_1)}$$

these terms do not affect the variation of S', since $q(t_1)$ & $q(t_2)$ are fixed

(2) consider now a primed frame moving very slowly
$\vec{V}_0 = \varepsilon \vec{V}_0$, $\varepsilon \to 0$

$$L(\vec{v}'^2) = L((\vec{v} - \epsilon\vec{V}_0)^2) = (\text{Taylor expand}) =$$
$$= L(\vec{v}^2 - 2\epsilon\,\vec{v}\cdot\vec{V}_0 + \epsilon^2 \vec{V}_0^2)$$
$$= L(\vec{v}^2) - 2\epsilon\,\vec{v}\cdot\vec{V}_0\,\frac{\partial L(\vec{v}^2)}{\partial|\vec{v}^2|} + \underline{O}(\epsilon^2)$$
$$= L(\vec{v}^2) - 2\epsilon\left(\frac{d}{dt}(\vec{r}\cdot\vec{V}_0)\right)\frac{\partial L(\vec{v}^2)}{\partial|\vec{v}^2|} + \underline{O}(\epsilon^2)$$

ONLY if $L(\vec{v}^2) = \text{const}\,\vec{v}^2$ is this t-independent

$$= L(\vec{v}^2) - \frac{d}{dt}\left(2\epsilon\,\vec{r}\cdot\vec{V}_0\,\frac{\partial L(\vec{v}^2)}{\partial|\vec{v}^2|}\right) + \underline{O}(\epsilon^2)$$

ONLY IF

so $L(\vec{v}^2) = \text{const}\,\vec{v}^2$

we have that $L(\vec{v}'^2) = L(\vec{v}^2) + \frac{d}{dt}(\text{---})$

for infinitesimal $\epsilon\vec{V}_0$.

? about finite V_0 ? $\longrightarrow$

— having shown that for small $\vec{V}_0$ Galileo's principle requires that $L(\vec{v}^2) = \text{const.}\ \vec{v}^2$, we need to show that it holds for any $\vec{V}_0$ ⟶

⟶ easy!

first pick $\text{const} = \frac{m}{2}$

so have $L(\vec{v}^2) = \frac{m}{2}\vec{v}^2 \equiv$ $\left(\in L(\vec{v}) = \frac{m}{2}\vec{v}^2\right)$

$$= L(\vec{v} - \vec{V}_0) = \frac{m}{2}\vec{v}^2 \underbrace{- m\,\vec{v}\cdot\vec{V}_0}_{\substack{\text{this is}\\ \frac{d}{dt}\left(-m\,\vec{r}\cdot\vec{V}_0\right)\\ \text{for any } \vec{V}_0}} + \underbrace{\frac{m}{2}\vec{V}_0^{\,2}}_{\text{this is constant}}$$

this is constant, so E.O.M unaffected (a constant shift of S does not matter)

⟶ so ⟶ homogeneity & isotropy of space & time
\+
Galileo's principle of relativity !

symmetry in action

⇓

$$L = \frac{m}{2}\vec{v}^2 \quad \text{for a free particle}$$

$m \equiv$ mass

$m > 0 \Leftrightarrow S$ is minimal for straight line motion from $1 \to 2$