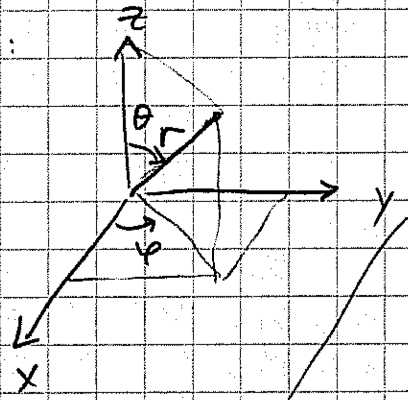
$$L = \frac{m}{2}\vec{v}^{2} = \frac{m}{2}\dot{\vec{r}}^{2}$$

$$\vec{r} = (x, y, z) \qquad \dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$$

so $L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$ [Cartesian]

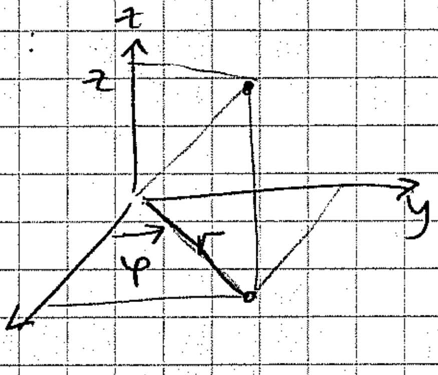
polar:



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \left(\text{do derivatives yourselfes}\right) = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$$

cylindrical:



$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\text{DIY})$$

$$= \dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2$$

For many particles (free)

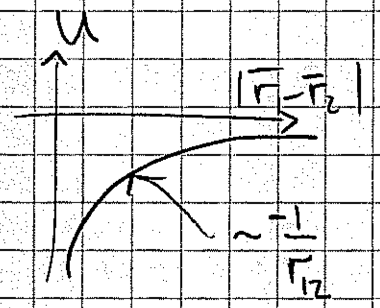
$$L = \sum_{a} \frac{m_a}{2}\dot{\vec{r}}_a^{\,2}, \qquad \vec{r}_a \text{ - position of a-th particle}$$

↑ # of particle

m_a - mass of a-th particle

If NOT free, the particles may interact via forces: i.e. every two particles of masses m_1, m_2 have

$$U_{12} = -G_N \frac{m_1 \; m_2}{|\vec{r}_1 - \vec{r}_2|} = U(|\vec{r}_1 - \vec{r}_2|)$$



as particles get closer $|\bar{r}_1 - \bar{r}_2| \downarrow$

$U_{12} \searrow$, hence attraction

(gravity is attractive)

hence

$$L = \sum_a \frac{m_a \dot{\vec{r}}_a^{\,2}}{2} - \underbrace{\sum_{a<b} U(|\vec{r}_a - \vec{r}_b|)}$$

kinetic energy of pairwise interaction – here assumed centrally symmetric, i.e. U depends on $|\vec{r}_1 - \vec{r}_2|$ NOT on direction — NOT always the case, i.e. if each particle is an electric dipole

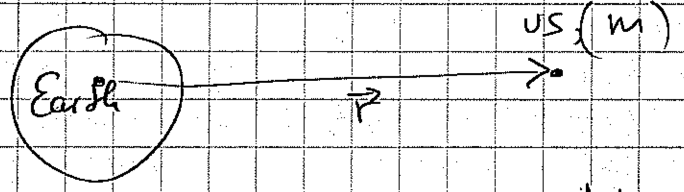
like atoms are ___

- also, in general can have non-pairwise interactions...

Generally, for our classical mechanics systems, we have

$$L = \underbrace{T}_{\Sigma \text{ of kinetic energies for all particles}} - \underbrace{U}_{\text{all potentials}}$$

Also, in classical nonrelativistic mechanics interactions are instanteneous:



$\vec{r}_\oplus = 0$

$$U = -G_N \frac{M_\oplus m}{r}$$

$$\vec{f} = -\vec{\nabla}_{\vec{r}} U \,, \qquad \vec{\nabla}_{\vec{r}} \frac{1}{r} = -\frac{1}{r^2} \vec{\nabla}_{\vec{r}} r$$

$$= -G_N M_\oplus m \frac{\vec{r}}{r^3} \qquad \qquad = -\frac{\vec{r}}{r^3}$$

so if Earth moves a bit (so $\vec{r}$ changes), the force changes instantaneously.

(no delay – even though gravity also propagates w/ c like E&M)

From the general

$$L = T - U$$

we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}_a} = \frac{\partial L}{\partial \vec r_a} \quad , a = 1 \ldots \# \text{ particles}$$

and hence $m\,\ddot{\vec r}_a = -\frac{\partial U}{\partial \vec r_a}$

$$\frac{\partial}{\partial \vec r_a} U(\vec r_1 \ldots \underset{\uparrow\,\#\text{ particles}}{\vec r_N}) = \underbrace{\left(\frac{\partial U(\vec r_1 - \vec r_N)}{\partial x_a}, \frac{\partial U(\vec r_1 - \vec r_N)}{\partial y_a}, \frac{\partial U(-)}{\partial z_a} \right)}_{(\text{a vector} \,\equiv\, \text{gradient of } U)}$$

Note: We argued that homogeneity, isotropy of space-time fixes (+ Galileo's relativity)

$$L_{free} = \frac{m\vec v^2}{2} , \ldots \qquad (\text{in inertial frame})$$

-- what about interactions?

-- what about time- & space- dependent potentials that we often consider?

* homogeneity of space $\overset{\text{\& time}}{\frown}$ means

interactions $U(\vec{r}_1 — \vec{r}_N)$

are (1) t - independent

(2) depend only on $\vec{r}_i - \vec{r}_j$ → so if the entire system of particles is moved by some constant $\vec{a}$, the interactions don't change

* isotropy of space?

⇒ U can only depend on <u>dot products</u>

($\vec{r}_1 \cdot \vec{r}_2$ is invariant under $\overset{\text{simultaneous}}{\frown}$ rotation of <u>both</u> particles)

e.g. for <u>N=2 only</u>: $|\vec{r}_1 - \vec{r}_2|^2 = (\bar{r}_1 - \bar{r}_2) \cdot (\bar{r}_1 - \bar{r}_2)$

for <u>N=3 more options:</u>

$$(\bar{r}_1 - \bar{r}_2) \cdot (\bar{r}_1 - \bar{r}_2)$$

potential can depend on these →

$$\left\{\begin{array}{l} (\bar{r}_1 - \bar{r}_2) \cdot (\bar{r}_1 - \bar{r}_3) \\ (\bar{r}_1 - \bar{r}_2) \cdot (\bar{r}_2 - \bar{r}_3) \\ (\bar{r}_1 - \bar{r}_3) \cdot (\bar{r}_2 - \bar{r}_3) \end{array}\right.$$

. . . etc.

(you count)

(which ones appear in U depends on physics outside mechanics - gravity, E&M, even QM, depending on situation at hand)

So, for a larger number of particles

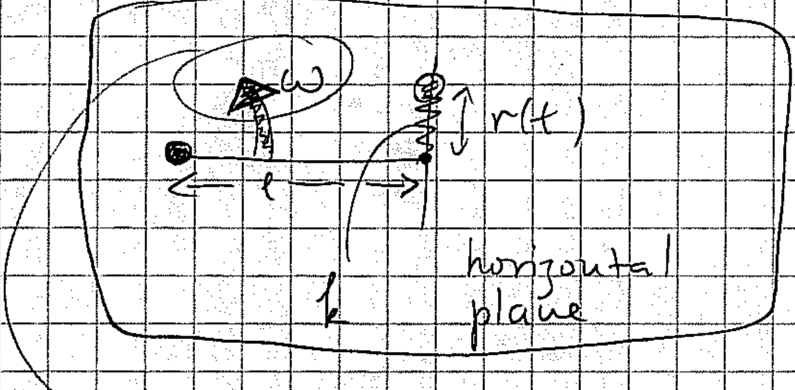
U can, in principle depend on many invariants.

———"———

Now very often we don't want to solve

E.O.M. for many $\begin{pmatrix}\text{even two!} \\ \text{or three}\end{pmatrix}$ particles.

— then we often consider a system as part of a closed system — as in your HW 1, #1



▷ this motion is prescribed, ω = const — you, or motor that rotates & keeps ω = const is not of interest, only r(t) matters.

— this leads to there being an "external" potential due to the rotation w/ ω — which makes it look like space is not homogeneous as U depends on r (as you'll see) — but of course if everything was translated, incl. motor, homogeneity is restored —

Finally a words about <u>constraints</u> —

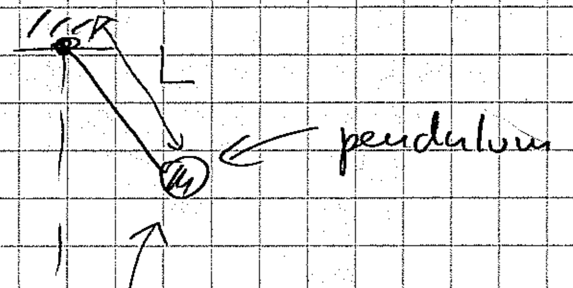
— many approaches — but we'll mostly consider constraints of the type (for N particles)

$$f_A(\vec{r}_1, \ldots, \vec{r}_N) = 0$$

some functions of positions, positions must obey this

$(f_1, f_2, \ldots)$

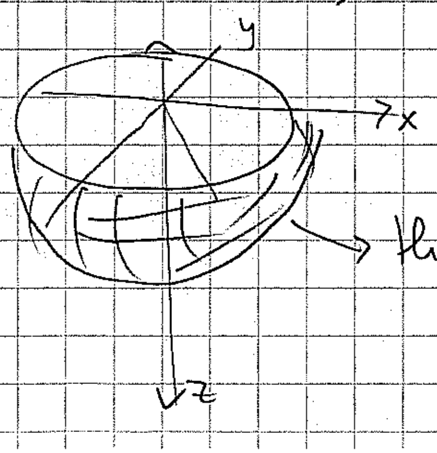
<u>For now, we'll use most common-sense approach:</u>



position of particle $\vec{r} = (x, y, z)$ in 3d

but L is fixed

so $\vec{r}^{\,2} = L^2$ is a constraint



this is a ½-sphere of radius L

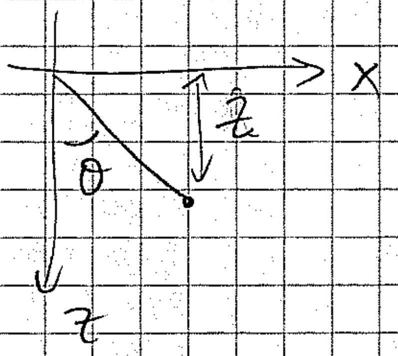
further, if allowed motion only in

z-x-plane, $y = 0$

→ so $y = 0$

$x^2 + y^2 + z^2 = L^2$ $\Big|$ $\Rightarrow$ $\underbrace{x^2 + z^2 = L^2}$

best to use angle θ



$x = L\sin\theta$

$z = L\cos\theta$

$$\dot{x}^2 + \underset{0}{\cancel{\dot{y}^2}} + \dot{z}^2 = L^2\left(\cos^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\theta}^2\right) =$$

$$= L^2\dot{\theta}^2$$

so $\begin{cases} T = \frac{m}{2}L^2\dot{\theta}^2 \\ U = -mg\underset{\updownarrow}{z} = -mgL\cos\theta \end{cases}$

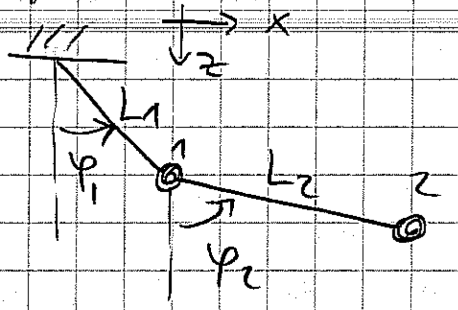
distance to x-axis ("−" sign: energy lower if h↑ → so want to fall ——)

$$L = \frac{m}{2}L^2\dot{\theta}^2 + mgL\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mL^2\ddot{\theta} = -mgL\sin\theta = \frac{\partial L}{\partial\theta}$$

$$\underline{\ddot{\theta} = -\frac{mg}{L}\sin\theta}$$ ⟷ (as we know.)

more complex ones, like



$x_1 = L_1 \sin\varphi_1$

$z_1 = L_1 \cos\varphi_1$

$x_2 = L_1 \sin\varphi_1 + L_2 \sin\varphi_2$

$z_2 = L_1 \cos\varphi_1 + L_2 \cos\varphi_2$

If in-plane need two angles only

$$\begin{cases} T = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{z}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{z}_2^2\right) \\ U_1 = -m_1 g z_1 - m_2 g z_2 \end{cases}$$

rest is algebra – constraints mean L_1 & L_2 do not depend on time, so only φ_1, φ_2 differentiate —▷ this suffices to do HW. –

there are other ways to deal w/ constraints, most notably the method of Lagrange multipliers –– –– later, perhaps

Symmetries, as you know by now are very important — we saw how they restricted L, for a free particle & for a system of interacting ones (which is closed)

Symmetries also imply conservation laws.

① homogeneity of time $\Rightarrow$ L for a closed system is t-independent

in general $L = L(q, \dot q, \cancel{t})$ for closed system $\Leftarrow$ homogeneity of time

Now

$$\frac{d}{dt}L = \sum_i \left(\frac{\partial L}{\partial q_i}\dot q_i + \frac{\partial L}{\partial \dot q_i}\ddot q_i + \underbrace{\cancel{\frac{\partial L}{\partial t}}}_{0}\right) \qquad (*)$$

Next, for a solution of the E-L equations, we have

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right), \quad \forall i$$

plug into $(*)$ $\rightarrow$

$$\frac{d}{dt}L = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\dot q_i + \frac{\partial L}{\partial \dot q_i}\overbrace{\ddot q_i}^{=\frac{d}{dt}\left(\dot q_i\right)} =$$

$$= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i\right)$$

$\Rightarrow$ put both terms on same side $\longrightarrow$

$$\frac{d}{dt}\left\{\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} q_i\right) - L\right\} = 0$$

→ since $\frac{d}{dt} = 0 \Rightarrow \{\ \} = \text{const}$, ie does not change in time if $(q, \dot{q})$ are actual trajectories, ie solve E-L equs.

We call: $\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} q_i\right) - L = E \equiv$ energy

hence that E is conserved (doesn't change in time) follows from homogeneity of time.

Qu.: is this E as we know it?

Ans: Yes.

let $L = \sum_i \frac{m_i}{2} \dot{q}_i^2 - U(\{q\})$

$$\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = (m_i \dot{q}_i) \cdot \dot{q}_i = m_i \dot{q}_i^2$$

so 1st term in E is

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_i m_i \dot{q}_i$$

so $E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \sum_i m_i \dot{q}_i - \left(\frac{1}{2} \sum_i m_i \dot{q}_i^2 - U\right) =$

$= \frac{1}{2} \sum_i m_i \dot{q}_i^2 + U = T + U$ as we know it

More generally, if q_i's are "generalized coordinates", we call

$$\frac{\partial L}{\partial \dot{q}_i} \equiv p_i \equiv \text{"generalized momenta"}$$

In general $\frac{\partial L}{\partial \dot{q}_i}$ depends on both q's & $\dot{q}$'s,

so $p_i = p_i(q, \dot{q})$ -

Ex: $L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r})$

$$\frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} = m\vec{v} = \vec{p}$$

(what we call "momentum" from 1st year)

→ in polar coordinates (x-y motion)

$$L = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\varphi}^2 - U(r) \quad \text{(say)}$$

$$\frac{\partial L}{\partial \dot{r}} = p_r = m\dot{r} \leftarrow \text{depends only on } \dot{r} \ (\dot{q}_i)$$

$$\frac{\partial L}{\partial \dot{\varphi}} = p_\varphi = mr^2\dot{\varphi} \longleftarrow \text{depends on } \underset{(\dot{q}_i)}{\dot{\varphi}} \ \& \ \underset{(q_i)}{\underline{\underline{r}}}$$

Normally, one's used to deriving energy conservation from Newton's equ:

$$m\ddot{x} = -U'(x) \quad \text{-- let } x \text{ obey this}$$

$\dot{x}$ | x above:

$$m\dot{x}\ddot{x} = -\dot{x}\,U'(x)$$

$$\Downarrow$$

$$\frac{1}{2}m\frac{d}{dt}(\dot{x})^2 = -\dot{x}\frac{d}{dx}U(x) = -\frac{d}{dt}\Big(U(x)\Big)$$

chain rule in reverse

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + U(x)\right) = 0 \longleftrightarrow \text{same as above.}$$

→ so a system whose L is t-independent conserves E.

② homogeneity of space ⇒ L for a closed system same if all coordinates (particles') shifted the same way:

($\vec{r}_i$ - position of i-th particle) $\vec{r}_i \to \vec{r}_i + \vec{a}$, $\vec{a}$: any vector.

let $L = L(\{\vec{r}_i\}, \{\dot{\vec{r}}_i\})$

$$L' = L(\{\vec{r}_i + \vec{a}\}, \{\dot{\vec{r}}_i + \underset{0}{\dot{\vec{a}}}\})$$

$\vec{a}$ is constant vector, t - independent

It is enough to consider an infinitesimal shift →

⇒ $\vec{a} = \epsilon \vec{b}$, ϵ - small

$$L' = L(\{\vec{r}_i + \epsilon \vec{b}\}, \dot{\vec{r}}_i) = L(\{\vec{r}_i\}, \{\dot{\vec{r}}_i\}) +$$

$$+ \epsilon \underbrace{\sum_i \frac{\partial L}{\partial \vec{r}_i} \vec{b}} + \underline{O}(\epsilon^2)$$

this should = 0 since $L' = L$ by homogeneity of space

$$\Rightarrow \quad \vec{b} \cdot \left(\sum_i \frac{\partial L}{\partial \vec{r}_i} \right) = 0$$

$\vec{b}$ is arbitrary - take $\left.\begin{array}{l} \vec{b} = (\hat{a}, 0, 0) \\ \vec{b} = (0, \hat{a}, 0) \\ \vec{b} = (0, 0, \hat{a}) \end{array}\right\}$ ⇒ all $\sum_i \frac{\partial L}{\partial \vec{r}_i}$ should = 0.

$$\Rightarrow \quad \sum_i \frac{\partial L}{\partial \vec{r}_i} = 0 \longrightarrow \text{by E-L we have}$$

$$\frac{\partial L}{\partial \vec{r}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_i}$$

hence

$$0 = \sum_i \frac{\partial}{\partial \vec{r}_i} L = \sum_i \frac{d}{dt}\frac{\partial}{\partial \dot{\vec{r}}_i} L$$

$$= \frac{d}{dt}\sum_i \frac{\partial L(\{\vec{r}_i\},\{\dot{\vec{r}}_i\})}{\partial \dot{\vec{r}}_i}$$

$$\rightarrow \quad \underbrace{\sum_i \frac{\partial L(\bar{r},\dot{\bar{r}})}{\partial \dot{\vec{r}}_i}}_{} = \text{const.}$$

$\sum_i \vec{p}_i = 0 \iff$ momentum conservation.

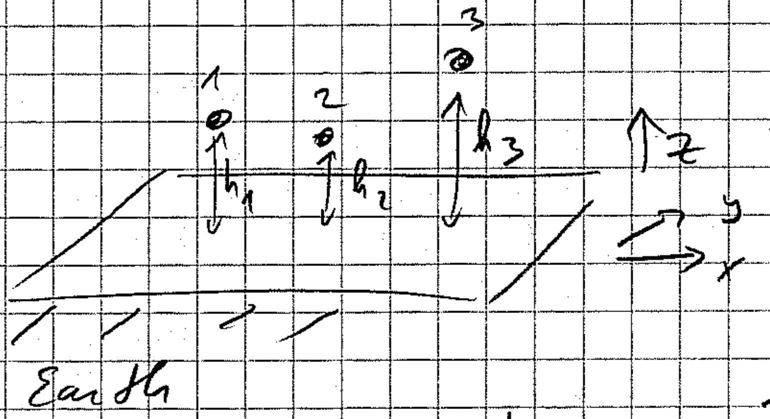
$$\vec{P} = \sum_i \vec{p}_i = \text{const.} = \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i}$$

⊛ total momentum = Σ individual particle momenta conserved due to homogeneity of space

[if there is an "external" potential, ie. if U depends NOT only on $\vec{r}_i - \vec{r}_j$, NOT conserved]

Comments :

(A) it may be that U does not depend on some coordinates only :



- momentum in $\hat{z}$ direction NOT conserved $U = m_1 g h_1 + m_2 g h_2 + m_3 g h_3$

- but $\hat{x}$ & $\hat{y}$ conserved, as the 3-particle system has translational invariance in the x-y plane (assuming, e.g. two body interactions, like rigid balls or so)

→ this follows the same derivation – translational invariance (= homogeneity of space) only holds in $\bar{x}$-$\bar{y}$, not $\bar{z}$

(— clear)

Ⓑ momentum conservation is the same as "<u>Newton's third law</u>" ⇒

⇒ translational inv'ce said

$$\underbrace{\sum_i \frac{\partial L}{\partial \vec{r}_i}} = 0 \quad \text{for our } L = \sum_i \frac{m}{2} \dot{\vec{q}}_i^{\,2} - U(\{q_i\})$$

$$\underbrace{\sum_i \left(- \frac{\partial U}{\partial \vec{r}_i} \right) = 0}$$

$\left(-\frac{\partial U}{\partial \vec{r}_i} = \text{force} \right)$ the sum of the forces acting on the particles of a closed system = 0

e.g. $\circ \xrightarrow{f_1}$ $\quad \xleftarrow{f_2} \circ$ $\rightarrow$ $\vec{f}_1 + \vec{f}_2 = 0$ ⇒

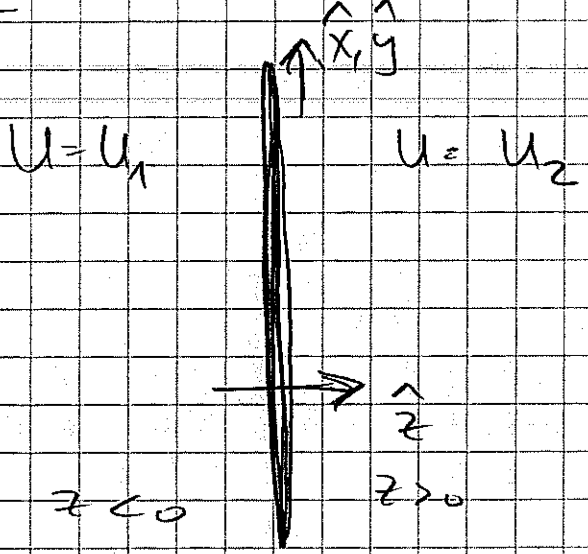
$$\Rightarrow \vec{f}_1 = -\vec{f}_2$$

more generally, $\dfrac{\partial L}{\partial q_i}$ = generalized forces

(recall $\dfrac{\partial L}{\partial \dot{q}_i}$ = generalized momenta)

† q_i — generalized coordinates

Ex: a particle moves in a potential which is

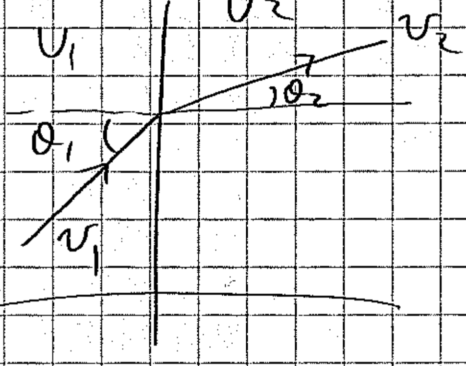


U_1 @ $z<0$ &
U_2 @ $z>0$
(otherwise constant) —

→ find change of direction upon crossing $z=0$.

- in general having U = const does not affect anything, as adding a constant to U does not change E-L equ.

- but here we have a region where U jumps, so $U_1 - U_2$ matters
- no $\hat z$ - translational invce
- but $\hat x, \hat y$ - invariance, so p_x & p_y conserved



* motion in-plane

* $v_1 \sin\theta_1 = v_2 \sin\theta_2$
($\hat x$-$\hat y$ momentum conservation)

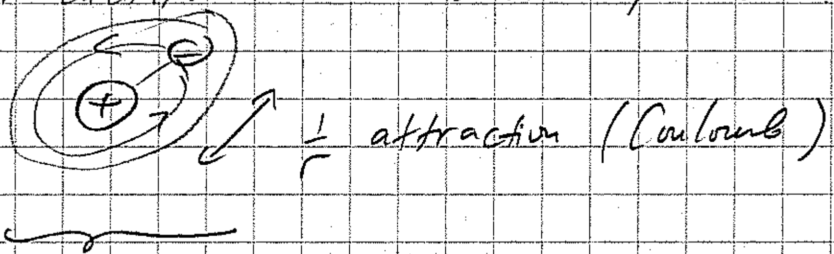
* $\frac{mv_1^2}{2} + U_1 = \frac{mv_2^2}{2} + U_2$
(energy conservation)

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{1+\frac{2}{mv_1^2}(U_1-U_2)}$$

USE OF CONSERV. LAWS !!

When describing a closed system of particles, it is often convenient to use a special coordinate system — the centre of mass system

i.e. atom / or Earth/Sun --- satellite/Earth

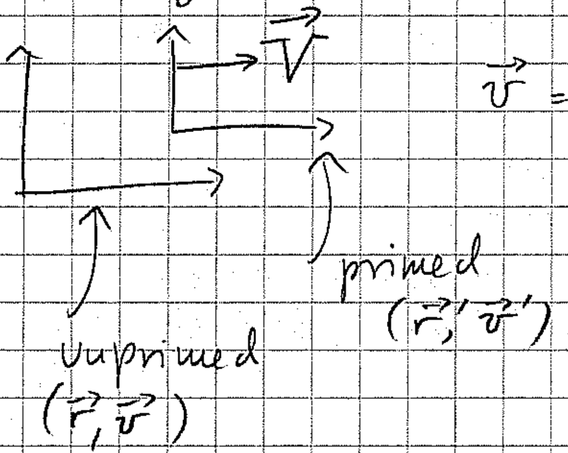


motion as a whole

instead of $\vec{r}_1, \vec{r}_2 \longrightarrow \vec{r}_{cm}, \vec{r}_{rel}$.

Also useful for a system of many particles (arbitrary number).

$\vec{v} = \vec{v}\,' + \vec{V}$, as we know



total momentum in unprimed system is $\vec{P} = \sum\limits_i m_i \vec{v}_i$

while in primed it is $\vec{P}\,' = \sum\limits_i m_i \vec{v}_i\,' =$

$$= \sum_i m_i \vec{v}_i - \sum_i m_i \vec{V} =$$

$$= \vec{P} - \vec{V}\Big(\sum_i m_i\Big) \longrightarrow$$

since $\sum_i m_i = \mu$ = total mass of system,

we have $$\vec{P} = \vec{P}' + \mu \vec{V} \qquad (**)$$

velocity of primed system wrt unprimed

((the primed system where total momentum = 0 is called the centre of mass system))

→ to find its velocity $\vec{V}_{CM}$ use (**)

& put $\vec{P}' = 0 \Longrightarrow \vec{V}_{CM} = \frac{\vec{P}}{\mu} = \frac{1}{\mu} \sum_i m_i \dot{\vec{r}}_i$

use def. of $\vec{P} = \sum_i m_i \vec{v}_i$

origin of CM system expressed i.t.o. coordinates of particles

$$\vec{V}_{CM} = \frac{d}{dt}\left(\frac{1}{\mu} \sum_i m_i \vec{r}_i\right)$$

$$= \frac{d}{dt} \vec{R}_{CM}.$$

$$\boxed{\vec{R}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \sum_i \frac{m_i}{\mu} \vec{r}_i}$$

(Ex: **N=2**: $\vec{R}_{CM} = \frac{m_1}{m_1+m_2} \vec{r}_1 + \frac{m_2}{m_1+m_2} \vec{r}_2$ — familiar? →

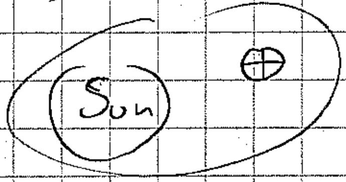
→ in particular, if $m_2 \gg m_1$,

we have

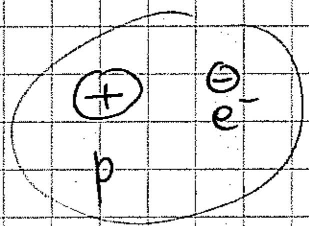
$$\vec{R}_{CM} = \underbrace{\frac{m_1}{m_1+m_2}}_{\ll 1} \vec{r}_1 + \underbrace{\frac{m_2}{m_1+m_2}}_{\approx 1} \vec{r}_2$$

$$\approx \vec{r}_2$$

e.g.



$\vec{R}_{CM} \approx \vec{r}_{Sun}$ (the $\overset{\text{much}}{}$ heavier)



$\vec{R}_{CM} \approx \vec{r}_p$ ($\overset{\text{since}}{m_p} \approx 2000\, m_e$)

If you have a closed system, its center of mass moves uniformly on a straight line – by homogeneity of space/time lagrangian can not depend on $\vec{R}_{CM}$.

Usually we use a system of reference where CM is at rest, i.e when describing (p^+e^-) or Sun – Earth, but motion of CM can be of interest, too.

the energy of a system of particles can also be written as:

(energy) = (energy of CM motion) + (internal energy)

$$E = \frac{1}{2}\mu \vec{V}_{CM}^{\,2} + E_{internal}$$

to show consider $E = \frac{1}{2}\sum_i m_i \vec{v}_i^{\,2} + U =$

$= \left(\text{use } \vec{v}_i = \vec{v}_i{}' + V_{CM}\text{, taking primed system} = CM\right) =$

$$= \frac{1}{2}\sum_i m_i\left(\vec{v}_i{}' + \vec{V}_{CM}\right)^2 + U =$$

$$= \frac{1}{2}\sum_i\left(m_i \vec{v}_i{}'^{\,2} + m_i \vec{V}_{CM}^{\,2} + 2 m_i \vec{v}_i{}' \cdot \vec{V}_{CM}\right) + U =$$

$$= \underbrace{\frac{1}{2}\left(\sum_i m_i\right)\vec{V}_{CM}^{\,2}} + \vec{V}_{CM}\left(\sum_i m_i \vec{v}_i\right) + \frac{1}{2}\sum_i m_i \vec{v}_i{}'^{\,2} + U$$

$$= \frac{1}{2}\mu \vec{V}_{CM}^{\,2} + \left(\vec{V}_{CM}\cdot\vec{P}'\right) + \left(\frac{1}{2}\sum_i m_i \vec{v}_i{}'^{\,2} + U\right)$$

but primed system = CM system

so $\vec{p}\,' = 0$

$$E = \underbrace{\frac{1}{2}\mu \vec{V}_{CM}^{\,2}}_{E_{\text{of CM}}} + \underbrace{\left(\frac{1}{2}\sum_i m_i \vec{v}_i{}'^{\,2} + u\right)}_{E_{\text{internal}}}\Bigg\}\text{ kinetic energy in CM + potential}$$

(50)

③ isotropy of space ≡ L doesn't change if a closed system is rotated as a whole

NOTE L&L talk about "isotropy of time", meaning really $t \to -t$ transform, which is more conventionally called "time reversal". This is a discrete, rather than continuous transformation.

$\ddot{x} = -\frac{1}{m} U'(x)$ invariant under $t \to -t$

meaning that, as opposed to translation (homogeneity of t & $\vec{r}$) or rotation, the notion of an infinitesimal time reversal does not make sense (while an infinitesimal rotation/translation is entirely intuitive - any translation/rotation can be thought as a sequence of many infinitesimal ones).

Discrete transforms do not lead to conservation laws. ⟶ such as T - time reversal

P - parity $\vec{r} \to -\vec{r}$

C - charge conjugation.

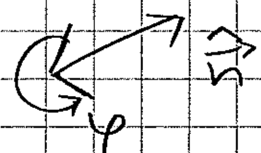
((CPT - yes

C, P & T separately NOT in weak interactions, yes in gravity, E&M ___))

So back to rotations of a closed system---

— what's a rotation?

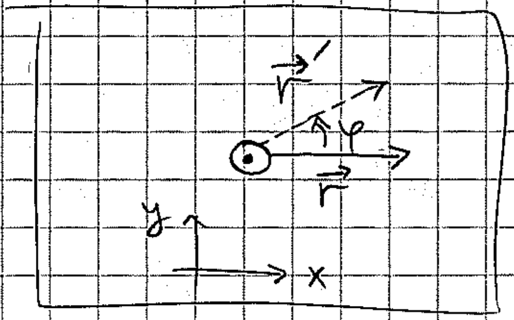
* need an axis $\equiv$ unit vector $\hat{\vec{n}}$



* need an angle

So there's always a plane in which one rotates $\equiv$

$\equiv$ the plane $\perp$ $\hat{\vec{n}}$



$\hat{\vec{n}}$ faces us (say along $\hat{z}$)

$\vec{r}\,'$ is $\vec{r}$ rotated by φ

let $\vec{r} = (x, 0, 0)$ so $|\vec{r}| = x$

then $\vec{r}\,' = (x\cos\varphi, x\sin\varphi, 0)$

Since $|\vec{r}\,'| = |\vec{r}|$: rotation doesn't change length of vectors

(call it $\delta\varphi$)

for small φ, we have

$$\vec{r}\,' \approx (x, x\delta\varphi, 0) + O(\delta\varphi^2) \quad (\delta\varphi \text{ - small} \to 0)$$

$$\vec{r} = (x, 0, 0)$$

it may look to you that $|\vec{r}'| \neq |\vec{r}|$

but NOTE that

$$|\vec{r}'| = \sqrt{x^2 + x^2\delta\varphi^2 + \dots} \approx x = |\vec{r}|$$

(remember $\varphi \to 0$)

to order φ, we neglect terms of order φ^2

hence $\vec{r}' - \vec{r} = \delta\vec{r} = (0, x\delta\varphi, 0)$

the change of $\vec{r}$ due to an infinitesimal rotation on $\delta\varphi$

this can be written as

$$\delta\vec{r} = \underbrace{(0, 0, \delta\varphi)}_{\delta\vec{\varphi}} \times \vec{r} = (0,0,\delta\varphi)\times(x,0,0)$$

$$(\delta\vec{r})_x = \overset{0}{\cancel{(\delta\vec{\varphi})_y}}(\vec{r})_z - (\delta\vec{\varphi})_z\overset{0}{\cancel{(\vec{r})_y}}$$

$$\longrightarrow \quad (\delta\vec{r})_y = (\delta\vec{\varphi})_z(\vec{r})_x - \overset{0}{\cancel{(\delta\vec{\varphi})_x}}(\vec{r})_z$$

$$(\delta\vec{r})_z = \overset{0}{\cancel{(\delta\vec{\varphi})_x}}(\vec{r})_y - \underset{0}{\cancel{(\delta\vec{\varphi})_y}}(\vec{r})_x$$

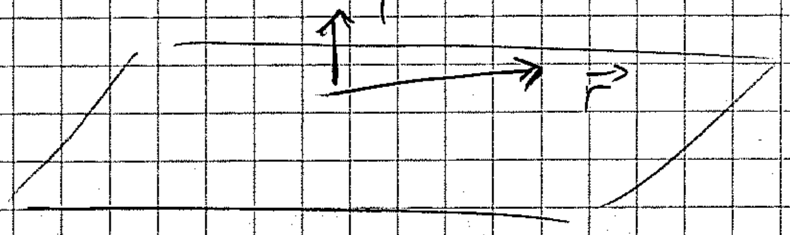
$$\boxed{\delta\vec{r} = \delta\vec{\varphi}\times\vec{r}}$$

– independent on coordinate system

(derived in a particular one, but more generally true!)

Equivalently, we can picture this like:

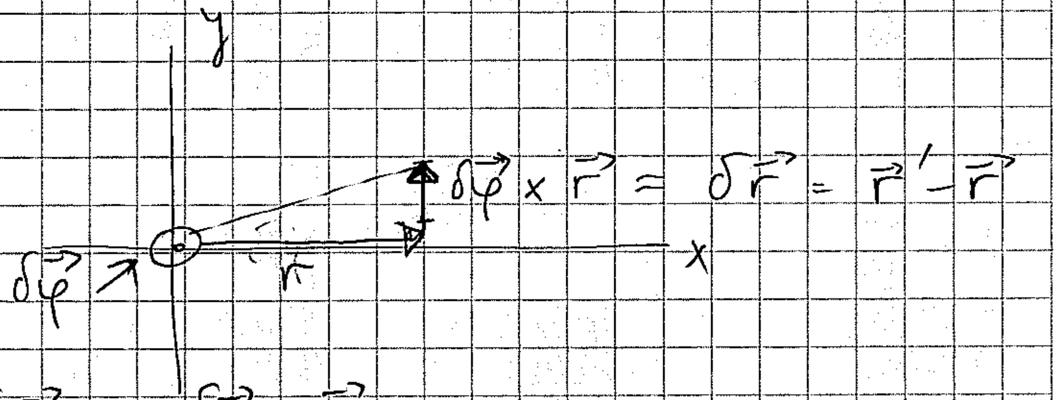
$\delta\vec{\varphi} \rightarrow$ a vector // $\hat{\vec{n}}$, whose length is $\delta\varphi$



$\delta\vec{\varphi} \times \vec{r}$ is a vector $\perp$ $\vec{r}$ & $\delta\vec{\varphi}$, i.e. in the x-y plane of length

$$|\delta\vec{\varphi}|\,|r|\,\underbrace{\sin(\text{angle between } \delta\vec{\varphi} \text{ \& } \vec{r})}_{\text{this} \equiv 1 \text{ since } \delta\vec{\varphi} \perp \vec{r}}$$

so from above



$$\delta\vec{r} \approx \delta\vec{\varphi} \times \vec{r}$$

$$\vec{r}\,' \approx \vec{r} + \delta\vec{r} = \vec{r} + \underbrace{\underbrace{\delta\vec{\varphi} \times \vec{r}}_{(0,\, x\,\delta\varphi,\, 0)}}_{\text{as we derived.}}$$

So, we have under a small rotation

$\delta\vec{\varphi}$ (direction $\equiv$ direction of rotation of $\delta\vec{\varphi}$

length of $\delta\vec{\varphi}$ $\equiv$ angle of rotatn)

$$\delta\vec{r} = \vec{r}\,' - \vec{r} \simeq \delta\vec{\varphi} \times \vec{r}$$

Same for velocities: (take $\frac{d}{dt}$, $\delta\vec{\varphi}$ is t-indep.)

$$\delta\vec{v} = \vec{v}\,' - \vec{v} \approx \delta\vec{\varphi} \times \vec{v} = \delta\vec{\varphi} \times \dot{\vec{r}}$$

We need $\delta\vec{r}$ & $\delta\vec{\varphi}$ in order to find

$$\delta L = L(\vec{r}\,', \vec{v}\,') - L(\vec{r}, \vec{v}) =$$

$$= \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot \delta\vec{r}_i + \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot \delta\dot{\vec{r}}_i =$$

$$= \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot (\delta\vec{\varphi} \times \vec{r}_i) + \underbrace{\frac{\partial L}{\partial \dot{\vec{r}}_i}}_{\equiv \vec{p}_i} \cdot (\delta\vec{\varphi} \times \dot{\vec{r}}_i)$$

change of L under an infinitesimal rotation of all particles

Now this should vanish, 'cause of isotropy of space

So we have

$$0 = \sum_i \frac{\partial L}{\partial \vec{r}_i} \cdot (\delta\vec{\varphi} \times \vec{r}_i) + \vec{p}_i \cdot (\delta\vec{\varphi} \times \dot{\vec{r}}_i) =$$

$$= \sum_i (\delta\vec{\varphi} \times \vec{r}_i) \cdot \frac{\partial L}{\partial \vec{r}_i} + \sum_i (\delta\vec{\varphi} \times \dot{\vec{r}}_i) \cdot \vec{p}_i =$$

$$= \left(\text{now: } (\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C}) \right)$$

$$= \delta\vec{\varphi} \cdot \left(\sum_i \left(\vec{r}_i \times \frac{\partial L}{\partial \vec{r}_i} + \dot{\vec{r}}_i \times \vec{p}_i \right) \right) = 0.$$

arbitrary vector — can rotate in any direction.

hence

$$\sum_i \left(\vec{r}_i \times \underbrace{\frac{\partial L}{\partial \vec{r}_i}} + \frac{d}{dt}(\vec{r}_i) \times \vec{p}_i \right) = 0$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i}\right) = \frac{d}{dt}\vec{p}_i \quad \left(\text{by E-L. } + p = \frac{\partial L}{\partial \dot{q}}\right)$$

$$\sum_i \vec{r}_i \times \frac{d}{dt}(\vec{p}_i) + \frac{d}{dt}(\vec{r}_i) \times \vec{p}_i = 0$$

$$\Downarrow$$

$$\sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = 0 \Rightarrow \frac{d}{dt}\left(\sum_i \bar{r}_i \times \bar{p}_1\right) = 0.$$

Again we have a conserved quantity:

$$\frac{d}{dt}\vec{M} = 0$$

$$\vec{M} = \sum_i \vec{r}_i \times \vec{p}_i$$ = the angular momentum of the closed system

(due to isotropy of space)

<u>We have</u>: $t \to t + \text{const}$ $E = \text{const}$

$$\left(E = \sum_i \dot{q}_i p_i - L \right)$$

$\vec{r}_i \to \vec{r}_i + \overrightarrow{(\text{const})}$ $\vec{P} = \text{const}$

$$\left(\vec{P} = \sum_i m_a \vec{v}_a = \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} = \sum_i \vec{p}_i \right)$$

$\vec{r}_i \to \vec{r}_i + \underset{(\text{const})}{\delta\vec{\varphi}} \times \vec{r}_i$: $\vec{M} = \text{const}$

$$\left(\vec{M} = \sum_i \vec{r}_i \times \vec{p}_i \right)$$

momentum & angular momentum are always <u>additive</u> $\equiv$

$\Rrightarrow$ total (angular) mom $\equiv \sum_i$ (individual ones) whereas energy is NOT additive –

– only for free particle systems.

(in general

$$E \neq \sum_i E_i$$).

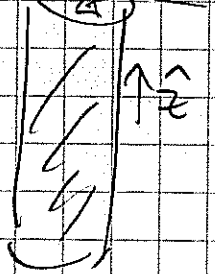
Of course, the values of E, $\vec{P}$, $\vec{M}$ depend on the frame. Transformation laws are easy to obtain (will do when/if needed).

Clearly $\vec{M}$ depends also on choice of origin of coordinates → under $\vec{r} \rightarrow \vec{r} + \vec{a}$

we have $\vec{M} \rightarrow \vec{M} + \vec{a} \times \vec{P}$

(HW! Show that $\vec{M} = \vec{M}_{CM} + \vec{R}_{CM} \times \vec{P}$)

As for translation case, a partial symmetry may also lead to angular momentum conservation along a particular direction ONLY → e.g.



an ∞ long mass (or charge)

– only rotations along $\hat{z}$ are symmetries

– hence only M_z conserved.