Mechanical similarity & virial theorem

* $L \to \text{const} \times L$ does not change E.O.M. (since $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ same)

$\underbrace{\phantom{L \to \text{const} \times L}}$ useful to deduce interesting things in case when potential is a homogeneous function.

homogeneous f-n: $f(x)$ - homogeneous of degree k

$$\text{if} \quad f(\alpha x) = \alpha^k f(x)$$

$$\left(f(x) = x^k \text{ - homogeneous of degree } k \right)$$

$f(x_1, x_2, \dots x_N)$ hom. of degree k if

$$f(\alpha x_1, \alpha x_2, \dots, \alpha x_N) = \alpha^k f(x_1 \dots x_N)$$

e.g.

$$f(x_1 \dots x_N) = x_1^{a_1} x_2^{a_2} \dots x_N^{a_N} + \text{similar terms w/ different } a\text{'s}$$

$$(a_1 + a_2 + \dots + a_N = k)$$

$$(\text{e.g. } x^3 + xyz + xy^2 + zx^2 \ \dots \text{ etc})$$

if $f(\alpha x) = \alpha^k f(x)$, let's take $\frac{d}{d\alpha}$ of this relation —

$$-\left(\frac{d}{d\alpha}\alpha^k = k\alpha^{k-1}\right) \Rightarrow \frac{df(\alpha x)}{d(\alpha x)} \frac{d(\alpha x)}{d\alpha} = k\alpha^{k-1} f(x)$$

$$\frac{df(\alpha x)}{d(\alpha x)} x = k\alpha^{k-1} f(x) \longrightarrow$$

put $\alpha = 1 \Rightarrow \underbrace{x \frac{df}{dx} = k f}$

Euler's theorem on homog. f-n.

if many variables:

$$f(\alpha x_1 \ldots \alpha x_N) = \alpha^k f(x_1 \ldots x_N)$$

same: $\left.\frac{d}{d\alpha}\right|_{\alpha=1}$ $x_1 \frac{\partial f(x_1 \ldots x_N)}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} f(x_1 \ldots x_N) +$

$$\ldots + x_N \frac{\partial}{\partial x_N} f(x_1 \ldots x_N) = k f(x_1 \ldots x_N)$$

$= U(\bar{x}_1 \ldots \bar{x}_N)$

— " —

Assume that $L = \sum_a \frac{m_a \vec{v}_a^2}{2} - \sum_{a<b} \underbrace{U(\vec{x}_a - \vec{x}_b)}$

homogeneous f-n

like $\frac{1}{|\bar{x}_a - \bar{x}_b|}$ or any other

we have

$$U(\alpha \vec{x}_1 \ldots \alpha \vec{x}_N) = \alpha^k \, U(\vec{x}_1 \ldots \vec{x}_N)$$

$k = -1$ (Coulomb, Newton, say)

rescale coordinates $\vec{x}_i$ by α } $\vec{v}_i$ rescaled by $\frac{\alpha}{\beta}$

rescale time t by β }

kinetic energy rescaled by $\frac{\alpha^2}{\beta^2}$ & U - by α^k

$\rightarrow$ but then, if we choose 75

$\frac{\alpha^2}{\beta^2} = \alpha^k$, the resulting rescaling

of $x \to \alpha x$, $t \to \beta t$ will be a rescaling of L by α^k

$\rightarrow \beta^2 = \alpha^{2-k} \rightarrow \beta = \alpha^{1-k/2}$

Hence for $U(\{\vec{x}\})$ homogeneous of degree k, we have that under $\vec{x}_a \to \alpha \vec{x}_a$

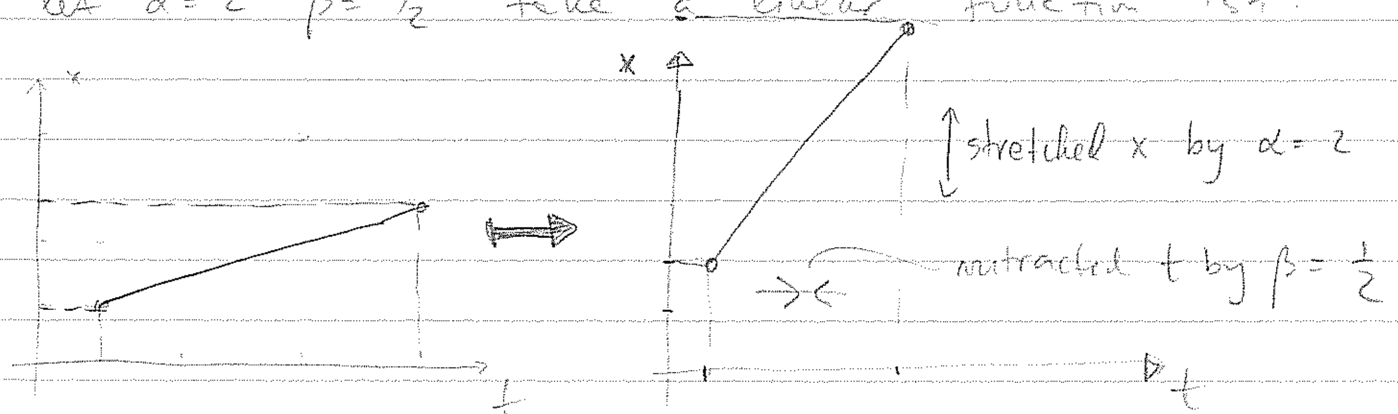
$t \to \alpha^{1-k/2} t$, $\mathcal{L} \to \alpha^k \mathcal{L}$

but this means – since EOM for $\mathcal{L}$ & $\alpha^k \mathcal{L}$ are identical – that if $\vec{x}(t)$ is a solution of EOM,

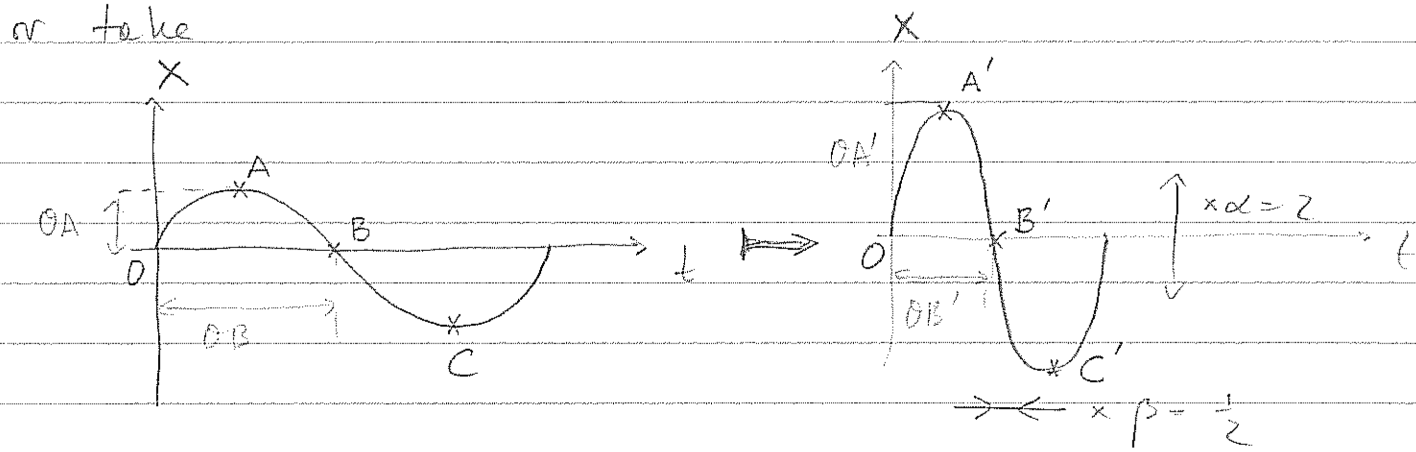
so $\underbrace{\vec{x}'(t) = \alpha \vec{x}(\beta t)}$, w/ $\beta = \alpha^{1-(k/2)}$

i.e. there is a whole family of geometrically similar trajectories

Ex: let $\alpha = 2$ $\beta = 1/2$ take a linear function 1st.



or take



this means that typical spatial size $\dfrac{OA'}{OA} = \alpha$

& typical times $\dfrac{OB'}{OB} = \beta = \alpha^{1-k/2}$

(as per $L \to L\alpha^k$)

$$\overset{\alpha}{} \qquad \left(\alpha^{\frac{2-k}{2}}\right)^{\frac{2}{2-k}}$$

& therefore $\dfrac{OA'}{OA} = \left(\dfrac{OB'}{OB}\right)^{\frac{2}{2-k}}$

or, more generally $\dfrac{l'}{l} = \left(\dfrac{t'}{t}\right)^{\frac{-2}{k-2}}$

or, as in book: $\boxed{\underbrace{\dfrac{t'}{t}}_{\text{times of motion between corresponding points on two similar paths}} = \left(\underbrace{\dfrac{l'}{l}}_{\text{ratio of linear dimensions of two paths}}\right)^{1-\frac{k}{2}}}$

from this relation, we can look @ how

$\frac{v'}{v}$ scales $\rightarrow \left(\text{since } v \sim \frac{\ell}{t} \right) \rightarrow \left(\frac{v'}{v}\right) = \left(\frac{\ell'}{\ell}\right) \frac{1}{\left(\frac{t'}{t}\right)} =$

$$= \frac{\frac{\ell'}{\ell}}{\left(\frac{\ell'}{\ell}\right)^{1-\frac{k}{2}}} = \left(\frac{\ell'}{\ell}\right)^{k/2}$$

so

$$\boxed{\frac{v'}{v} = \left(\frac{\ell'}{\ell}\right)^{\frac{k}{2}}}$$

while energies' ratio is $\sim \frac{v'^2}{v^2} = \left(\frac{\ell'}{\ell}\right)^{k}$ $\left(E = \underbrace{\frac{mv^2}{2} + U}_{\text{same scaling}} \right)$

& angular momentum is $\sim \left(\frac{\ell'}{\ell}\right)\left(\frac{v'}{v}\right) =$ $(M = |\bar{r} \times \bar{p}|)$

$$= \left(\frac{\ell'}{\ell}\right)\left(\frac{\ell'}{\ell}\right)^{k/2} = \left(\frac{\ell'}{\ell}\right)^{1+\frac{k}{2}}$$

$$\frac{v'}{v} = \left(\frac{\ell'}{\ell}\right)^{k/2}$$

$$\frac{E'}{E} = \left(\frac{\ell'}{\ell}\right)^{k}$$

$$\frac{M'}{M} = \left(\frac{\ell'}{\ell}\right)^{1+k/2}$$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{k}{2}}$$

$\rightarrow$ What does this imply?

(1) let $U(x) \sim x^2$ (harmonic oscillator, any dimension)

$\Downarrow$

$k = 2$

for two similar trajectories, we find that

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{2}{2}} = 1 \Rightarrow$$

$\Rightarrow$ $\underbrace{\text{period}}_{\text{(time between corresponding points)}}$ independent on $\underbrace{\text{amplitude}}_{\text{(size of trajectory)}}$

(2) let $\underbrace{U = -Fx}$ (homogeneous $\vec{E}$-field, or homogeneous gravity field)

$$k = 1$$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\frac{1}{2}} = \sqrt{\frac{\ell'}{\ell}}$$

e.g. $\left(\text{ratio of two times to fall}\right) \sim \sqrt{\text{ratio of heights from which one falls}}$

(3) let $U \approx \frac{1}{r}$, $k = -1$

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1+\frac{1}{2}} = \left(\frac{\ell'}{\ell}\right)^{\frac{3}{2}}$$

$$\left(\frac{t'}{t}\right)^2 = \left(\frac{\ell'}{\ell}\right)^3$$

$\left(\text{ratio of periods of two orbits}\right)^2 \sim \left(\text{ratio of radius of orbit}\right)^3$

(Kepler's 3rd law)

(4) "Virial theorem"

kinetic energy T is a homogeneous f-n of $\vec{v}_a$ of order 2

$\left(T = \sum_a \frac{m_a \vec{v}_a^{\,2}}{2}\right.$, clearly so$\left.\right)$

so, by Euler's theorem on p. 74 we have

$$\sum_a \underbrace{\vec{v}_a \cdot \frac{\partial T}{\partial \vec{v}_a}}_{= \vec{p}_a} = 2T$$

(assuming $L = T - U$) (indep. of $\vec{v}_a$)

we have $2T = \sum_a \vec{v}_a \cdot \vec{p}_a =$

$$= \sum_a \left(\frac{d}{dt}(\vec{p}_a \cdot \vec{r}_a) - \left(\frac{d}{dt}\vec{p}_a\right) \cdot \vec{r}_a \right)$$

$$= \frac{d}{dt}\left(\sum_a \vec{p}_a \cdot \vec{r}_a\right) - \sum_a \dot{\vec{p}}_a \cdot \vec{r}_a$$

"Lemma":

let's average w.r.t. time, for any $f(t)$, $\bar{f} \equiv \lim_{\tau \to \infty} \int_0^\tau dt\, f(t) \Big/ \tau$

also if $f(t) = \frac{dF(t)}{dt}$, & $F(t)$ is bounded, we have

$$\bar{f} = \lim_{\tau \to \infty} \int_0^\tau dt \frac{dF(t)}{dt} \Big/ \tau = \underbrace{\lim_{\tau \to \infty} \frac{F(\tau) - F(0)}{\tau} \to 0}$$

since $F(\tau)$ is bounded

Now, we assume that the system we're looking at is executing bounded motion (e.g. no particles $\to \infty$, ever $\Rightarrow$ e.g. only closed trajectories, no ∞ ones) then we look at

$$2T = \frac{d}{dt}\left(\sum_a \vec{p}_a \vec{r}_a\right) - \sum \dot{\vec{p}}_a \cdot \vec{r}_a$$

and let's average it wrt time (i.e. average both sides)

since finite motion & finite energy, both $\vec{p}_a$ & $\vec{r}_a$ are $< \infty$, $\forall a$, hence the $\frac{d}{dt}$ term averages to $\emptyset$.

so we have

$$2\overline{T} = -\overline{\sum_a \dot{\vec{p}}_a \vec{r}_a}, \quad \text{now } \dot{\vec{p}}_a = \frac{\partial U}{\partial \vec{r}_a}$$

$$\underbrace{2\overline{T} = \overline{\sum_a \frac{\partial U}{\partial \vec{r}_a} \cdot \vec{r}_a}}$$

this is always true — only used that $T \sim v^2$ —

if U is homogeneous of degree k, then by Euler theorem we know

$$\sum_a r_a \frac{\partial U}{\partial r_a} = k U$$

so we have

$$\underline{2\overline{T} = k\overline{U}}$$

this is called the "virial th-m"

it says that the time averages of T & U in a potential homogeneous of degree k are simply related; since $\bar{E} = E = \bar{T} + \bar{U}$ we can also say $E = \bar{T} + \bar{U} = \frac{k}{2}\bar{U} + \bar{U}$

$$\rightarrow \quad \bar{U} = \frac{2}{k+2} E$$

$$\text{or} \quad E = \bar{T} + \frac{2}{k}\bar{T} \quad \text{so}$$

$$\bar{T} = \frac{k}{2+k} E$$

Ex: $k = 2$ (harmonic osc.) $\quad \bar{U} = \bar{T} = \frac{1}{2}E$

$k = -1$ (Newton, Coulomb) $\quad \bar{U} = 2E$

$\bar{T} = -E$ (total $E<0$ for finite motion)

one application of this fact →

→ equipartition in stat. mech ⇒ we know $\overline{\frac{m\vec{v}^2}{2}} = \frac{3}{2} k T_{emper.}$

(statistical average = time average in the equilibrium system)

$$\bar{T}_x = \overline{\frac{m v_x^2}{2}} = \frac{1}{2} k T_{emperature}$$

so if x is a vibrational direction

$$\bar{T}_x = \bar{U}_x = \overline{\frac{m\omega^2 x^2}{2}} = \frac{1}{2} k T_{emperature} \text{ as well}$$

Another ex. of scaling, not involving homogeneous potentials →

two particles, same U (same ext. force) w/ different mass

$$\underbrace{m\ddot{X}}_{} = f$$

$$\left.\begin{array}{l} m = \alpha m \\ t = \sqrt{\alpha}\, t \end{array}\right\} \text{ leaves l.h.s invt.}$$

so if $m_1 = \alpha\, m_2$

$$t_1 = \sqrt{\alpha}\, t_2$$

$$\left(\frac{t_2}{t_1}\right) = \frac{1}{\sqrt{\alpha}} = \left(\frac{m_2}{m_1}\right)^{1/2}$$

so if (1) is $4\times$ heavier than (2)
it will pass same orbit (in the same field)
$2\times$ slower than (2)

---

back to analysis of central-field motion.