Kepler's problem finally →

recall that

(p. 65)

$$\begin{cases} E = \dfrac{m\dot{r}^2}{2} + \dfrac{M^2}{2mr^2} + U(r) \\ M = mr^2\dot{\varphi} \end{cases}$$

2 cons. laws
2 d.o.f.
⇓
(great!)

$$\begin{cases} d\varphi = \dfrac{M}{mr^2}\,dt \\ dr = \sqrt{\dfrac{2}{m}\left(E - U(r)\right) - \dfrac{M^2}{m^2r^2}}\,dt \end{cases}$$

→ solve any problem, in principle (Mathematica/Maple)

- start w/ given M, E
- find $\underbrace{r_{min} \ (\& \ r_{max}}$, if motion is bound)

  * the solutions of $\frac{2}{m}(E - U(r)) = \frac{M^2}{m^2r^2}$ equation, i.e. the turning points

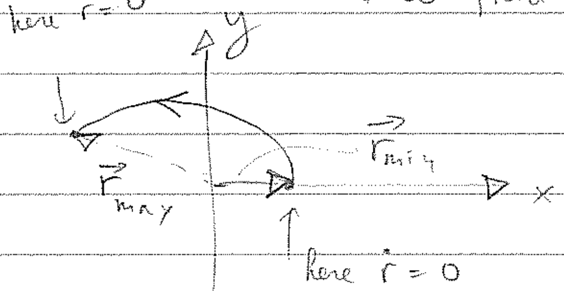
- integrate, say, from r_{min}: (in Mathematica [NIntegrate[---]])

$$\begin{cases} \dfrac{dr}{dt} = \sqrt{\dfrac{2}{m}\left(E - U(r)\right) - \dfrac{M^2}{m^2r^2}} \\ r(0) = r_{min} \\ \cancel{\dot{r}(0) = 0} \ \text{(velocity = 0 @ turning point)} \end{cases}$$

– find $r(t)$ between r_{min} & r_{max}

use to solve
$$\begin{cases} \dfrac{d\varphi}{dt} = \dfrac{M}{m r^2(t)} \\ \varphi(0) = 0 \quad (\text{measure angle from } r_{min}, \text{ that is}) \\ \cancel{\dot{\varphi}(0) = \dfrac{M}{m r_{min}^2}} \end{cases}$$

→ to find $\varphi(t)$ between r_{min} & r_{max}



– then the entire trajectory follows by reflecting this piece along the $0 - \vec{r}_{max}$ axis until $|r^{new}_{min}|$ reached, then reflect along $0 - \vec{r}^{\,new}_{min}$ axis etc. ...

(recall in general trajectory won't close)

Now, "luckily", for $1/r$ we can integrate:

Dividing two eqns on p.(83) we have, for $U = -\dfrac{\alpha}{r}$ (attractive)

$$\frac{d\varphi}{dr} = \frac{M}{m r^2} \frac{1}{\sqrt{\frac{2}{m}\left(E + \frac{\alpha}{r}\right) - \frac{M^2}{m^2 r^2}}}$$

$$d\varphi = \frac{M}{m} \frac{dr}{r^2} \frac{1}{\sqrt{\frac{2}{m}\left(E + \frac{\alpha}{r}\right) - \frac{M^2}{m^2 r^2}}} \qquad \text{let } \frac{1}{r} = u, \ \frac{dr}{r^2} = -du$$

Now, $$d\varphi = -\frac{M\,du}{\left(2mE + 2m\alpha u - M^2u^2\right)^{1/2}}$$

$$= -\frac{du}{\left(\frac{2mE}{M^2} + \frac{2m\alpha}{M^2}u - u^2\right)^{1/2}}$$

hence $$\varphi = \varphi_0 - \int \frac{du}{\sqrt{\frac{2mE}{M^2} + \frac{2m\alpha}{M^2}u - u^2}}$$

Now: $$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{-c}}\cos^{-1}\left(\frac{-b-2cx}{\sqrt{b^2-4ac}}\right)$$

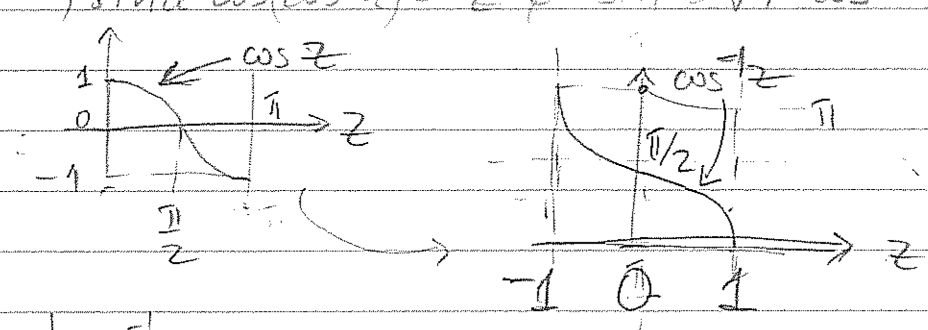
Proof: $\frac{d}{dx}(\text{rhs})$ = integrand in l.h.s

recall $\cos(\cos^{-1}z) = z$ so $\frac{d}{dz}\cos(\cos^{-1}z) = 1$, but

this is $\underbrace{-\sin(\cos^{-1}z)}_{\sqrt{1-z^2}}\frac{d}{dz}\cos^{-1}z = 1$

$\sqrt{1-z^2}$, since $\cos(\cos^{-1}z) = z$ & $\sin = \sqrt{1-\cos^2}$.

in pictures



hence $-\sqrt{1-z^2}\,\frac{d}{dz}\cos^{-1}z = 1$

$$\rightarrow \frac{d}{dz}\left(\cos^{-1}z\right) = -\frac{1}{\sqrt{1-z^2}}$$ match w/ picture

then, indeed

$$\frac{d}{dx}\left(\frac{1}{\sqrt{-c}}\cos^{-1}\left(\frac{-b-2cx}{\sqrt{b^2-4ac}}\right)\right)=$$

$$= -\frac{1}{\sqrt{-c}}\frac{1}{\left(1-\frac{(b+2cx)^2}{b^2-4ac}\right)^{1/2}}\frac{-2c}{(b^2-4ac)^{1/2}} =$$

$$= 2\sqrt{-c}\frac{1}{\sqrt{b^2-4ac-(b+2cx)^2}} =$$

$$= 2\sqrt{-c}\frac{1}{\sqrt{\cancel{b^2}-4ac-\cancel{b^2}-4c^2x^2-4bcx}} =$$

$$= \frac{1}{\sqrt{a+bx+cx^2}} \quad \blacksquare$$

---

then we have $a=\frac{2mE}{M^2}$, $b=\frac{2m\alpha}{M^2}$, $c=-1$

$$x \to u$$

& $$\varphi = \varphi_0 - \cos^{-1}\left(\frac{-\frac{2m\alpha}{M^2}+2u}{\left[\left(\frac{2m\alpha}{M^2}\right)^2+4\left(\frac{2mE}{M^2}\right)\right]^{1/2}}\right)$$

and

$$\cos^{-1}\left(\frac{\frac{2}{r}-\frac{2m\alpha}{M^2}}{\left[\left(\frac{2m\alpha}{M^2}\right)^2+4\left(\frac{2mE}{M^2}\right)\right]^{1/2}}\right) = \varphi_0-\varphi$$

& so

$$\frac{\cancel{2}}{r} - \frac{\cancel{2}m\alpha}{M^2} = \left(\left(\frac{\cancel{2}m\alpha}{M^2}\right)^2 + \cancel{4}\left(\frac{2mE}{M^2}\right)\right)^{1/2} \cos(\varphi - \varphi_0)$$

$$\boxed{\frac{1}{r} = \frac{m\alpha}{M^2}\left(1 + \sqrt{1 + \frac{2EM^2}{m\alpha^2}}\right)\cos(\varphi - \varphi_0)}$$

$\left(\cos\varphi - \varphi_0 = = \cos\varphi_0 - \varphi\right)$

denote: $p = \frac{M^2}{m\alpha}$ — dim of distance ($V = -\frac{\alpha}{r}$ energy)

("latus rectum" of orbit)

$e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$ — dim. less "eccentricity"

so we have

$$\underbrace{\frac{p}{r} = 1 + e\cos\varphi}$$

, let's put $\varphi_0 = 0$ (no loss of generality)

what is this? — equation of trajectory $\vec{r}(r, \varphi)$ coordinates — —

(A) S'ppose $e > 1$, i.e. $E > 0$

then $\left(\frac{p}{r}\right)_{min} = 1 - e \Rightarrow r_{max} = \infty$, because $e > 1$ & $1-e$ can vanish

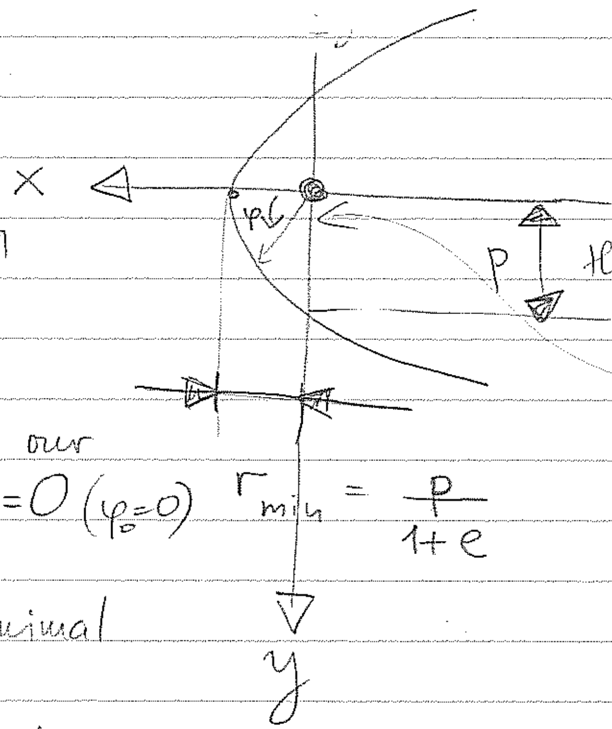
$\left(\frac{p}{r}\right)_{max} = 1 + e \to r_{min} = \frac{p}{1+e}$ distance of closest approach

here, of course, since $r > 0$, we can only reach angles such that $e|\cos\varphi| < 1$ or $|\cos\varphi| < \frac{1}{e} < 1$

since ($E > 0 \to e > 1$)

Note: φ is bound by $1 + e\cos\varphi \geq 0$
in fact $\cos\varphi \geq -\frac{1}{e}$

in $e > 1$ $(E > 0)$ we then have



this is r @ $\varphi = \frac{\pi}{2}$: $r = p$

center of potential

recall in our coord. $\varphi = 0$ $(\varphi_0 = 0)$ is where r is minimal here

$$r_{min} = \frac{p}{1+e}$$

(hence weird x–y direction – but orientation is as it should be)

(Note also that this is a hyperbola –

Proof: $\frac{p}{\sqrt{x^2+y^2}} = 1 + e\frac{x}{\sqrt{x^2+y^2}} \Rightarrow p = \sqrt{x^2+y^2} + ex$

$$\Rightarrow p - ex = \sqrt{x^2+y^2}$$

(more on p.88.)

$$\Rightarrow p^2 - 2epx + e^2x^2 = x^2 + y^2$$

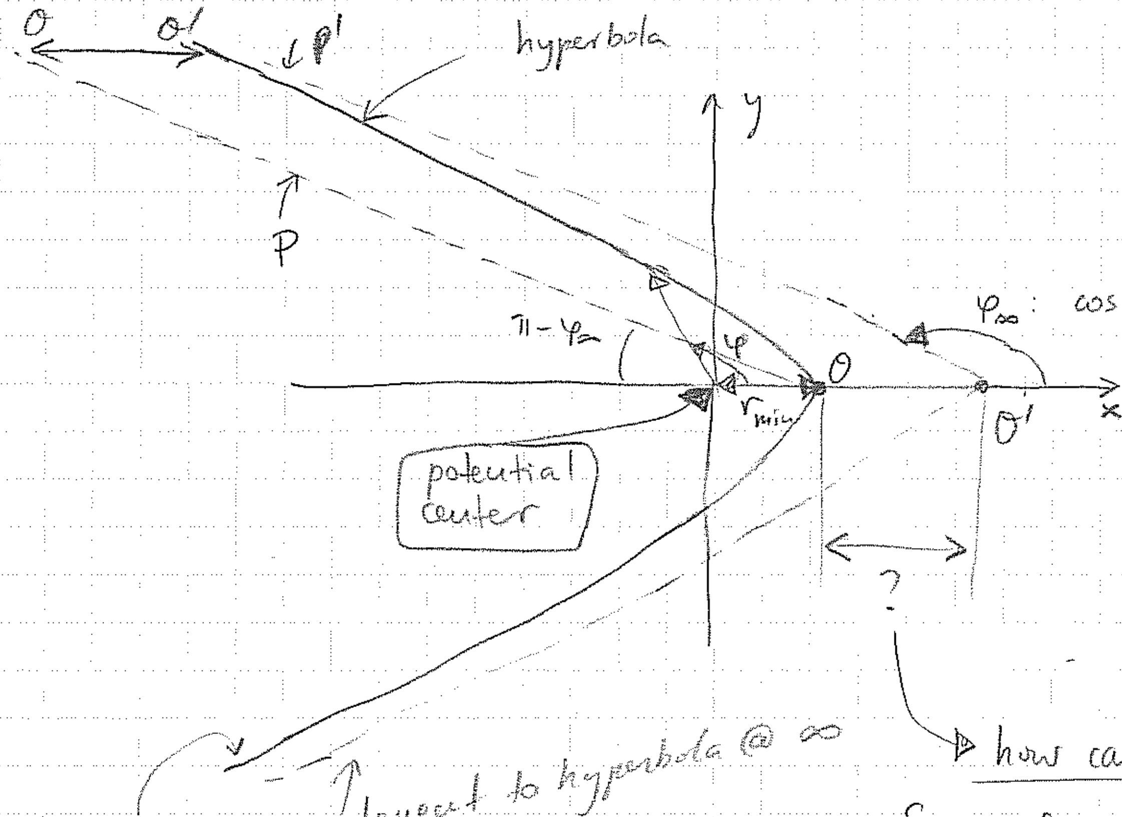
$$y^2 = p^2 - 2epx + \underbrace{(e^2-1)}_{>0,\ e>1}x^2$$

So, @ $e > 0$ $(E > 0)$ & $M \neq 0$ we have ∞ motion.

? about $e < \cancel{1}$ $(E < 0?)$

88.1

If we draw picture of p (88) in "normal" $\hat{x}-\hat{y}$ orientation, we have:



φ_∞: $\cos\varphi = -\frac{1}{e}$ (min value of φ s.t. $\frac{1}{r} \geq 0$.)

how can we find this?

$\frac{p}{r} = (1 + e\cos\varphi)$

Eqn. of OP line is

$y = x\tan(\pi - \varphi_\infty) + r_{min}$

OO' is distance from OP to hyperbola along x-axis in limit $(y, x) \to \infty$ ✱

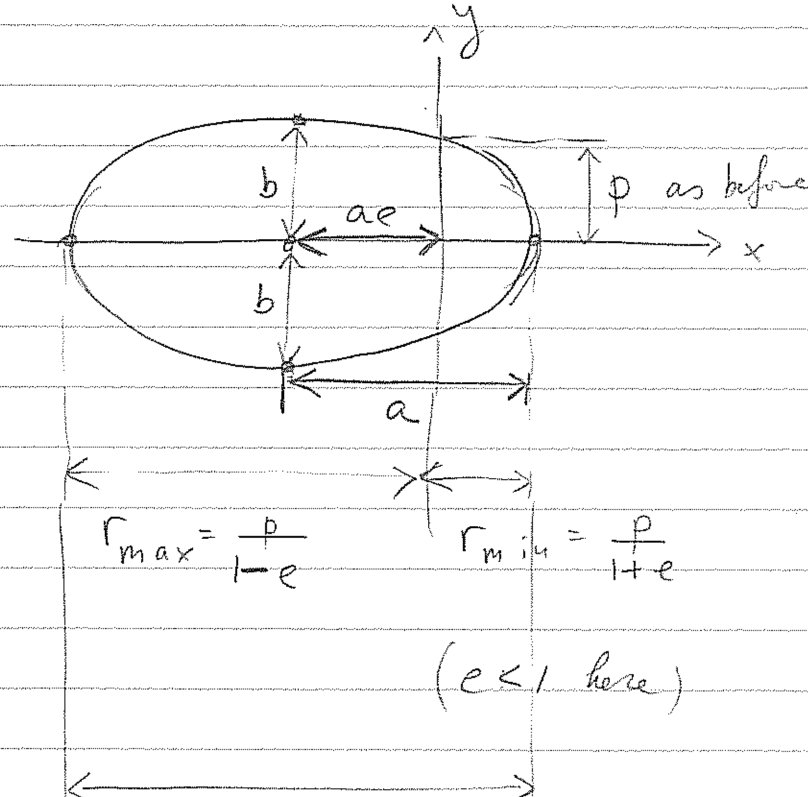
B $\quad e < 1, \qquad E > 0$

$$\left(\frac{p}{r}\right)_{min} = \frac{p}{r_{max}} = 1 - e \qquad (\varphi = \pi)$$

$$\left(\frac{p}{r}\right)_{max} = \frac{p}{r_{min}} = 1 + e \qquad (\varphi = 0)$$

$r_{max} = \dfrac{p}{1-e}$ at $\varphi = \pi$

$r_{min} = \dfrac{p}{1+e}$ at $\varphi = 0$



$b = ?$ occurs where $\dfrac{dy}{dx} = 0$

$$y = \pm\sqrt{p^2 - (1-e^2)x^2 - 2epx}$$

$\dfrac{dy}{dx} = 0$ when $(1-e^2)x = -ep$

so $x_* = -\dfrac{e}{1-e^2}p =$

$= -ae$

$2a = r_{max} + r_{min} = \dfrac{2p}{1-e^2} \equiv 2a$

$$\left(a \equiv \frac{p}{1-e^2}\right)$$

then

$$y(x_*) = \pm\sqrt{p^2 - (1-e^2)\frac{e^2}{1-e^2}p^2 + \frac{2e^2p^2}{1-e^2}}$$

$$y(x_*) = \pm\frac{p}{\sqrt{1-e^2}}$$

so $b = \dfrac{p}{\sqrt{1-e^2}} \longrightarrow \dfrac{a}{b} = \dfrac{1}{\sqrt{1-e^2}} \Rightarrow 1 - e^2 = \dfrac{b^2}{a^2} \rightarrow$

$\rightarrow e = \sqrt{1 - \dfrac{b^2}{a^2}} =$

e = eccentricity = 1 for circle

$$= \sqrt{1 - \left(\frac{\text{short semiaxis}}{\text{long semiaxis}}\right)^2} < 1 \text{ for ellipse}$$

$$= \sqrt{1 - \frac{2|E|M^2}{m\alpha^2}}$$ (written for ECO)

—— // ——

historically, motion in $1/r$ $\longleftrightarrow$ planets,

Kepler's laws (now we know all three —

* elliptical orbits
* $\left(\frac{T'}{T}\right)^2 = \left(\frac{L'}{L}\right)^3$
* constant sectorial velocity
  $\left(\left(\frac{\text{area}}{\Delta t} = \text{const}\right)\right)$)

deviations from $1/r$

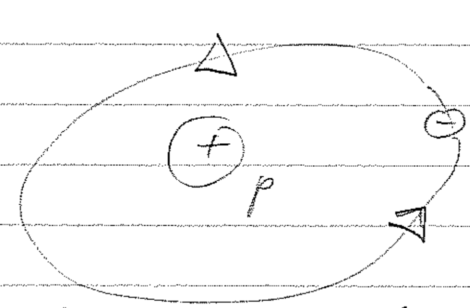
* other planets
* non sphericity of Sun, say
* General relativity
  (not $V(r)$ $\Leftrightarrow$ geodesics in Schwarzschild metric)

$\to$ but also Coulomb is $1/r$

— atoms $\longrightarrow$

we can use the notion of classical trajectory

to describe



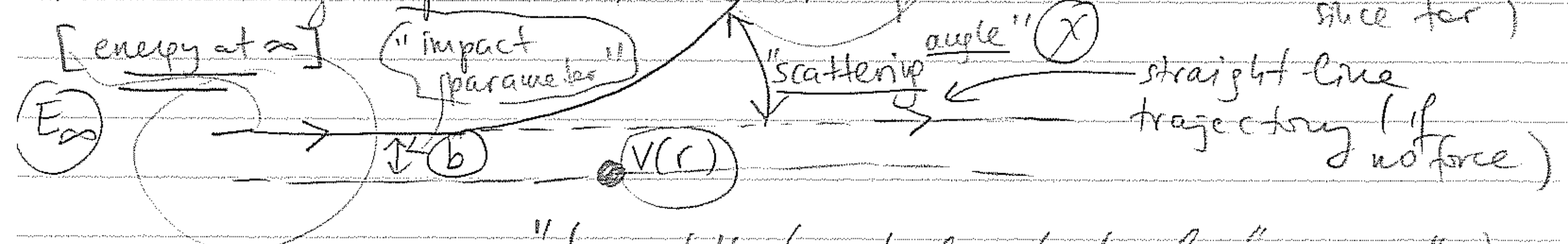
in $\frac{1}{r}$ Coulomb

potential, but (except for orbits with large angular momentum — large action $\gg \hbar$) — we'll not be in great agreement w/ experiment since QM effects matter [eg electrons radiate —]

But another problem which can be solved in CM and is relevant is that of SCATTERING ≡

≡ deviation of a particle moving in the field of another particle from the straight-line motion (due to potential, of course);

For a single particle, we imagine



So things that matter are

(1) energy of scattering particle @ ∞; $\underline{E_\infty}$ (>0)
(2) impact parameter $\underline{b}$ (in book called "p")
(3) form/strength of potential, say $V(r)$
(4) scattering angle χ

— in atomic (solid state) physics can control

$\underbrace{\underline{E_\infty, b} \text{ and measure } \underline{\chi(E,b)}}$

→ learn about $V(r)$?

("probe")

— in most cases, we scatter a beam of particles off a target

I = intensity of beam = flux of beam = (in book called "n")

= # particles going thru unit area ⊥ to the beam in unit time

Now, we observe that dN particles pass through a detector – far away, so particles again free – covering a solid angle $d\Omega$, in unit time

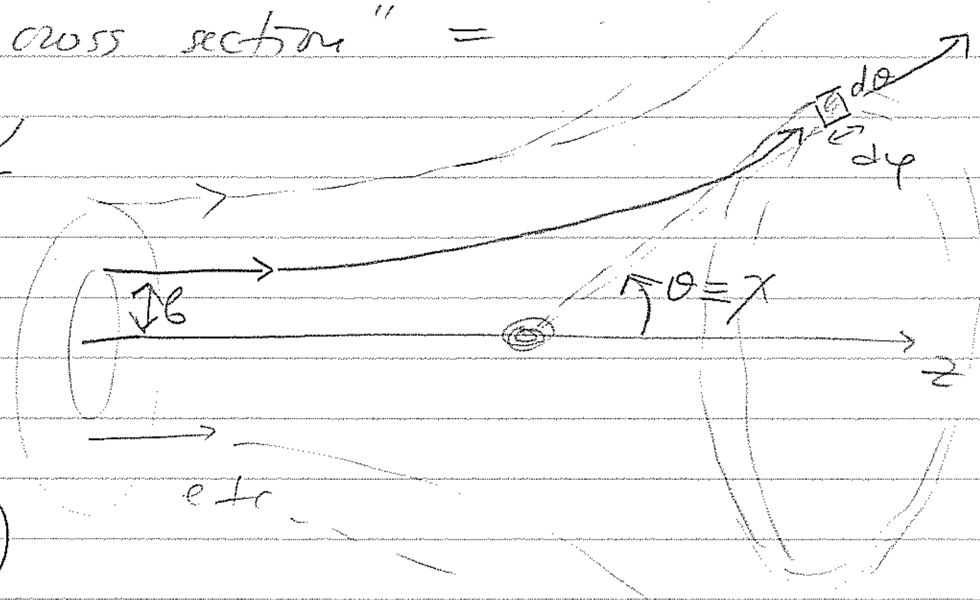
Of course dN is $\sim I$ (the more the flux the more particles scatter) but we know I (since we create it in lab)

so makes sense to divide dN / I

$\underbrace{dN/I}$ this is indep. of intensity (flux)

"differential cross section" =

$$\Rightarrow d\sigma = \frac{dN}{I}$$



(usually we assume that all incoming particles have same E_∞)

$d\Omega = 2\pi \sin\chi \, d\chi$ — assume cylindrical symmetry ($\theta = \chi$)

$$\left(\left(\int d\Omega = \int_0^{\pi} 2\pi \sin\chi \, d\chi = -2\pi \cos\chi \Big|_0^{\pi} = 4\pi \quad \text{as it should} \right)\right)$$

We need to express χ thru b (and energy, of course).

But we know the solution, at least formally!

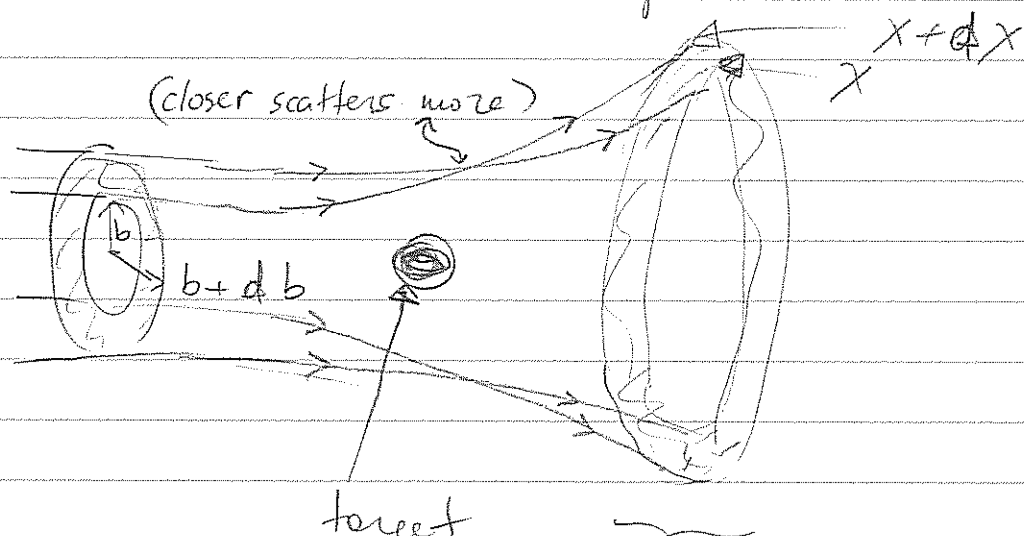
@ ∞ we have $E = \dfrac{m v_\infty^2}{2}$, $M = m v_\infty b$, v_∞ - speed @ ∞

(we'll assume that all beam particles have same v_∞)

Further, let $V(r)$ be such that the relation $\chi(b)$ is 1-1 $\Longleftrightarrow$ meaning that if a particle scattered by χ, one knows for sure which b it came from [true in Coulomb, repulsive — but in some attractive potentials - NOT! (—hm—)]

what can happen is that several b's can give same χ

then:



in $d\Omega = 2\pi\chi\, d\chi$, then,

we have, in unit time, that all particles in the beam that crossed the b, b+db area will be scattered

(* steady state process
* beam width "infinite")

So

$$dN = \underbrace{2\pi b\, db}_{\text{area of}} \cdot \underbrace{I}_{\text{flux}} \Rightarrow d\sigma = \frac{dN}{I} = 2\pi\, b(\chi)\, db(\chi)$$

we want it all expressed thru χ, of course — since that's where the info is:

$$d\sigma = 2\pi\, b(\chi) \left| \frac{d b(\chi)}{d\chi} \right| d\chi \quad \text{, of course}$$

since sign doesn't matter
usually $b'(\chi) < 0$
since increasing b decreases χ
(common sense)!

$$= \frac{b(\chi)}{\sin\chi} \left| \frac{d b(\chi)}{d\chi} \right| \underbrace{2\pi \sin\chi\, d\chi}_{d\Omega}$$

(*) if $b(\chi)$ has several branches, must include $\sum$ over them

$$\frac{d\sigma}{d\Omega} = \frac{b(\chi)}{\sin\chi} \left| \frac{d b(\chi)}{d\chi} \right| \rightarrow \text{differential cross section}$$

- characterizes scattering process, not beam intensity — — depends on interaction — range, sign, r-dependence ——
- this is given in "lab" frame — target at rest

$$\sigma_{total} = \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^{\pi} 2\pi \sin\chi\, d\chi\, \frac{b(\chi)}{\sin\chi} \left| \frac{d b(\chi)}{d\chi} \right|$$

"total cross section"

— Now $\Rightarrow$ $\boxed{\left[\dfrac{d\sigma}{d\Omega}\right]} = \dfrac{\text{area}}{\text{solid angle}}$

$$[\sigma] = \text{area}$$

$$\underbrace{\frac{d\sigma}{d\Omega}}_{} \times I \times T = \underbrace{\begin{pmatrix}\text{\# scattering events} \\ \text{in given } d\Omega\end{pmatrix}}_{}$$

I: $\underbrace{\dfrac{\text{\# particles}}{\text{1 area 1 time}}}$

T: $\underbrace{\text{length of time of experiment}}$ — could be year(s)

prediction for # events of a given type.

theory prediction (form of interactions --> --> "Standard Model" or "Beyond")

"luminosity" LHC $10^{34} \frac{1}{cm^2 s}$ (hopefully) (ask the LHC engineers)

is my theory right?

((lots of activity — Higgs, supersymmetry, technicolor - - — we don't use CM to find $\frac{d\sigma}{d\Omega}$, but big picture is same!))

but, back to Earth, all of this was used (first?) by Rutherford —

let look @ α particles off nuclei [or generally, "$-Z'e$"-charged scattering particles off a center of charge "$-Ze$"]

at rest

* $V(r) = \dfrac{ZZ'e^2}{r}$, repulsive, so $\underbrace{\alpha = -ZZ'e^2}_{\text{in our notn of (pp. 84-90)}}$

* we have $E > 0$, since ∞ trajectories,

and so $\dfrac{1}{r} = \dfrac{mZZ'e^2}{M^2}\left(-1 + e\cos\varphi\right)$ ← p 87, top formula, be careful when $\alpha \to -\alpha$

came from this → $\left[\dfrac{1}{r} = \dfrac{\alpha m}{M^2}(1 + e\cos\varphi)\right]$; $e = \sqrt{1 + \dfrac{2EM^2}{m(ZZ'e^2)^2}}$

$$M = mv_\infty b$$

$$E = \frac{mv_\infty^2}{2}, \quad mv_\infty = \sqrt{2Em}$$

so $M^2 = (mv_\infty b)^2 = b^2\, 2Em$

so $e = \sqrt{1 + \dfrac{2E\, b^2\, 2Em}{m(ZZ'e^2)^2}} = \sqrt{1 + \dfrac{4E^2b^2}{(ZZ'e^2)^2}}$

$$e = \sqrt{1 + \left(\frac{2Eb}{ZZ'e^2}\right)^2}$$ ← express thru E, b & $(Z, Z', e^2) = |\alpha|$

$e > 1$ since $E > 0$ & we have ∞ trajectory how does this look on the plane?

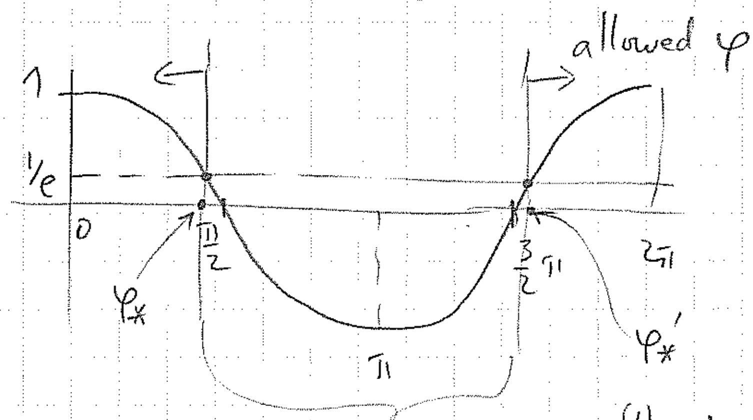
we have $\underbrace{\frac{p}{r}}_{\text{always} \geq 0} = \underbrace{-1 + e\cos\varphi}$ hence $-1 + e\cos\varphi \geq 0$

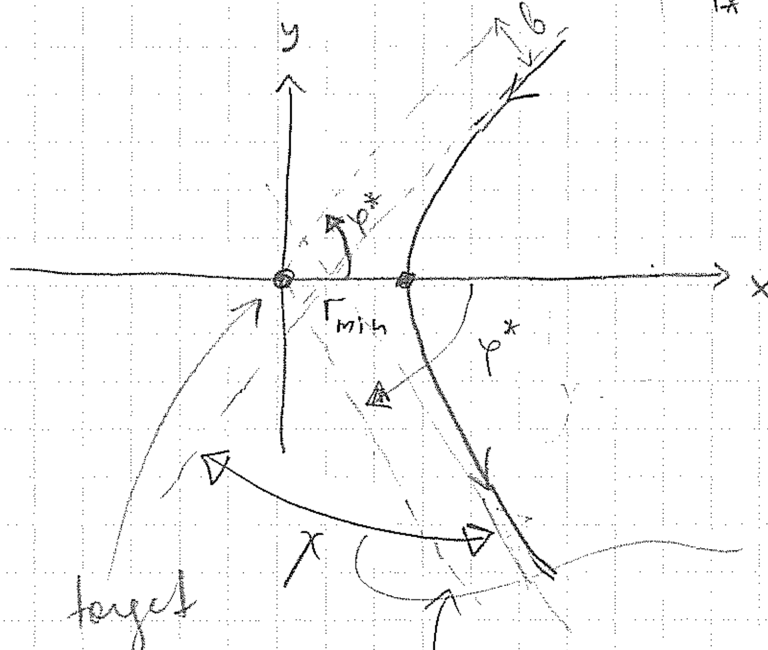
$\Leftrightarrow \quad \cos\varphi \geq \frac{1}{e}$

r_{min} is where

$e\cos\varphi - 1$ is maximal,

i.e. $\varphi = 0$: $\quad r_{min} = \frac{p}{e-1}$





scattering angle ; $\chi = \pi - 2\varphi_*$

orbit should be // to these lines (tangential to some lines // to these) @ ∞

So we know $\quad \cos\varphi_* = \frac{1}{e} = \frac{1}{\sqrt{1 + \left(\frac{2Eb}{ZZ'e_{el.m}}\right)^2}}$

Solve for $b \rightarrow$

$$\frac{1}{\cos^2\varphi_*} = 1 + \left(\frac{2Eb}{zz'e_{elm}}\right)^2 \quad \text{so}$$

$$\left(\frac{2Eb}{zz'e_{elm}}\right)^2 = \frac{1}{\cos^2\varphi_*} - 1 = \frac{1-\cos^2}{\cos^2} = \tan^2\varphi_*$$

$$b^2 = \left(\frac{zz'e_{elm}}{2E}\right)^2 \tan^2\varphi_*$$

but $\varphi_* = \frac{\pi - \chi}{2}$ (picture)

$$b^2 = \left(\frac{zz'e_{elm}}{2E}\right)^2 \tan^2\left(\frac{\pi}{2} - \frac{\chi}{2}\right) \qquad \tan\frac{\pi}{2} - x = \cotan\frac{\pi}{2}$$

$$= \left(\frac{zz'e}{2E}\right)^2 \cotan^2\frac{\chi}{2} \quad ; \text{ now, } \frac{d\sigma}{d\Omega} = \left|\frac{1}{\sin\chi}\frac{b\,db}{d\chi}\right|$$

so, need, by diff. b^2 wrt χ:

$$2b\frac{db}{d\chi} = \left(\frac{zz'e}{2E}\right)^2 \frac{d}{d\chi}\left(\frac{\cos^2\frac{\chi}{2}}{\sin^2\frac{\chi}{2}}\right)$$

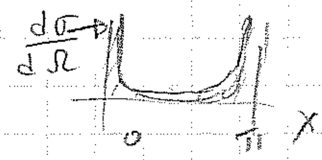
$$= \left(\frac{zz'e}{2E}\right)^2 \frac{2\cos\chi/2\,\sin\chi/2}{\sin^4\chi/2}\,\frac{1}{2}$$

$$\left|b\frac{db}{d\chi}\right| = \left(\frac{zz'e}{2E}\right)^2 \frac{1}{4}\,\frac{\sin\chi}{\sin^4\frac{\chi}{2}}$$

famous Rutherford formula

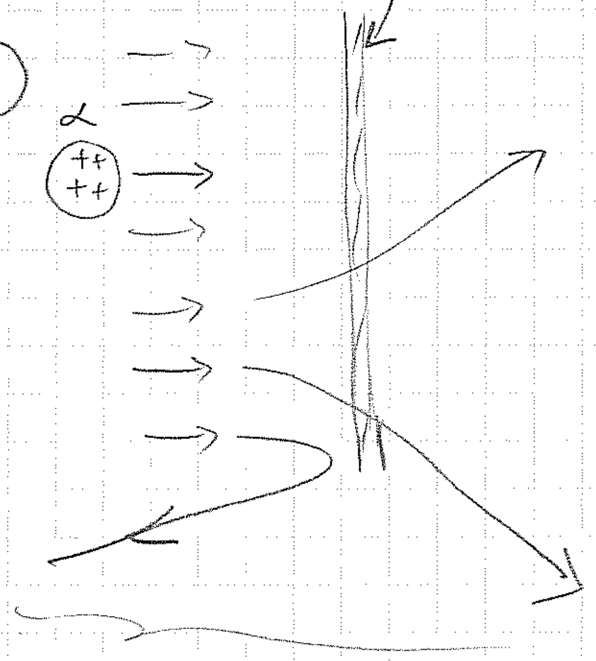
$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\left(\frac{zz'e}{2E}\right)^2 \frac{1}{\sin^4\chi/2} = \left(\frac{zz'e}{4E}\right)^2 \frac{1}{\sin^4(\chi/2)}$$

$$\frac{d\sigma}{2\pi \sin\chi \, d\chi} = \left(\frac{ZZ'e}{4E}\right)^4 \frac{1}{\sin^4 \frac{\chi}{2}}$$



in Non.Rel. QM same applies for Coulomb, exactly!

gold (very thin, so ≈ one or a few atomic layers)



1909-11 observed many hard scatters inconsistent w/ "gelee" model of Thompson

smeared "+" charge

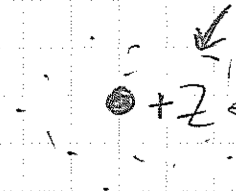
electrons

neutral atom

but instead

electrons

(ignorable for E_α he used — α's get close to nucleus)



nucleus
strong repulsion

estimated size of nucleus:

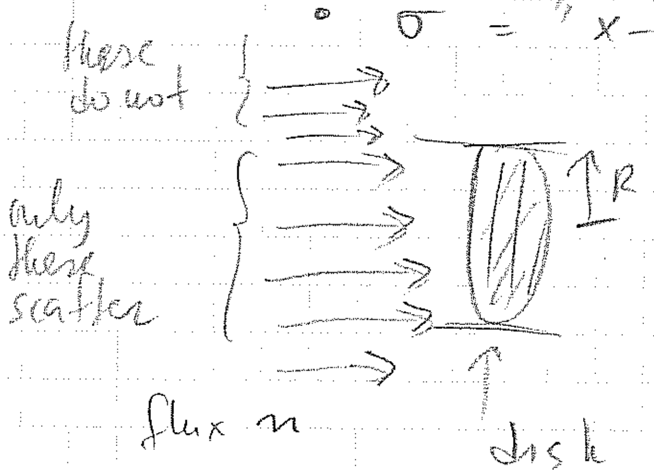
* E_α known, Z_α, Z_{Au} known →

→ estimate r_{min} by $E = \frac{ZZ'}{r_{min}} e^2$ → got 10^{-14} m, close (10^{-15} really) his stuff still useful! —

What about σ_{total}? → for Rutherford, diverges if one takes χ integral — $\frac{d\sigma}{d\Omega}$ blows up at small χ

- very generic in CM — for a long-range potential total σ is always ∞, since all particles scatter, even those w/ $b \to \infty$ (since long range)

- σ = "x-section" = $\left(\dfrac{\text{total \# scattering events per unit time}}{\text{flux}} \right)$



$= \pi R^2$

- $\sigma_{total}^{Rutherford} = \infty \iff R = \infty \to$ all scatter.

- in QM, even for long range $\sigma_{total} < \infty$, since uncertainty relation limits $\chi_{scattering}$ — can't be too small $b\chi > \hbar$, roughly speaking