(d.1)

let $L(\vec{r}_1 \ldots \vec{r}_N, \dot{\vec{r}}_1 \ldots \dot{\vec{r}}_N) =$

$$= \sum_i \frac{m_i \dot{\vec{r}}_i^{\,2}}{2} - \sum_{i \neq j} U(\vec{r}_i - \vec{r}_j)$$

<u>Galileo:</u>

$$\vec{r}_i \longrightarrow \vec{r}_i + \vec{V}_0 t$$

$$\dot{\vec{r}}_i \longrightarrow \dot{\vec{r}}_i + \vec{V}_0$$

symmetry $\equiv$ action invariant

not necessarily L!

i.e. $\underbrace{L(\{\vec{r} + \vec{V}_0 t,\ \dot{\vec{r}} + \vec{V}_0\})}_{} - L(\{\vec{r}, \dot{\vec{r}}\}) =$

$$= \sum_i m_i \dot{\vec{r}}_i \cdot \vec{V}_0 =$$

$\equiv \frac{\vec{R}_{CM}}{M}$

as shown in class

$$= \frac{d}{dt}\left[\left(\sum_i m_i \vec{r}_i\right) \cdot \vec{V}_0\right]$$

l.h.s

for small $\vec{V}_0$

r.h.s, for any $\vec{V}_0$, including small

$$= \sum_i \frac{\partial L}{\partial \vec{r}_i} \vec{V}_0 t + \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \vec{V}_0 =$$

$$||$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}_i}\right) \vec{V}_0 t + \frac{\partial L}{\partial \dot{\vec{r}}_i} \frac{d}{dt}(\vec{V}_0 t) = \qquad \frac{d}{dt}\left(\frac{\vec{R}_{CM}}{\mu} \cdot \vec{V}_0\right)$$

$$= \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot \vec{V}_0 t\right) = \frac{d}{dt}\left(\vec{P} \cdot \vec{V}_0 t\right)$$

d.2

so Gallilean invce ⇒

$$\frac{d}{dt}\left(\frac{\vec{R}_{CM}}{\mu}\cdot\vec{v}_0 - \vec{P}t\cdot\vec{v}_0\right) = 0$$

$$\Rightarrow \frac{\vec{R}_{CM}}{\mu}\cdot\vec{v}_0 - \vec{p}t\cdot\vec{v}_0 = \vec{R}_0/\mu\cdot\vec{v}_0$$

constant vector $\times\ \vec{v}_0$

$$\Rightarrow \boxed{\vec{R}_{CM} = \frac{\vec{P}}{\mu}t + \vec{R}_0}$$

conserved quantity due to Gallilean transf's.

Moral: Noether theorem ⟷ action invariant (not necessarily L)

for Gallileo — → CM moves on straight line

C.0

Claim: d d.o.f.

$\longleftrightarrow$ at most $2d-1$ independent $\int$ of motion. (f-ns of q, $\dot q$ that are t-indep.)

$2d$ = # initial conditions

d q's + d $\dot q$'s (arbitrary)

- but trajectory dep on t_0 as well — one can trade t_0 for one of arbitrary constants
- t_0 can not be fixed by $\int$ of motion

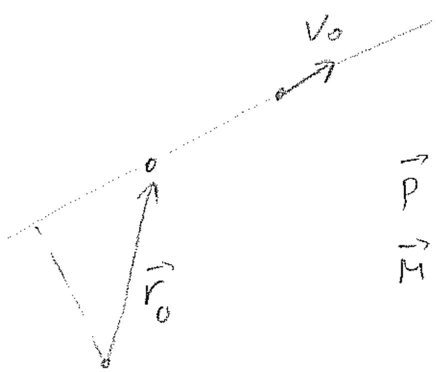
$$\vec r = \vec r_0 + \vec v_0 (t - t_0)$$

$$= (\vec r_0 - \vec v_0 t_0) + \vec v_0 t$$

shift t_0 ≡ shift of $\vec r_0$ ≈ choice of $\vec v_0$

So at most $2d-1$ indep. $\int$ of motion.

free. E, $\vec P$, $\vec M$ $\left\{ \begin{array}{l} E = \vec P^2/2m \\ \vec P \cdot \vec M = 0 \end{array} \right\}$ $\underline{5!}$



$\vec P \Leftrightarrow \vec v_0$

$\vec M \Leftrightarrow$ only $\vec r_0 \perp$ to $\vec v_0$

If "d" - integrable

"$>d$" - superintegrable

"$2d-1$" max. superint.

Comments on uses of symmetry ...

generally : symmetries $\Rightarrow$ conservation laws

$\downarrow$

use to simplify
solving the problem
(since these quantities are
guaranteed not to change –
– whatever they are @
beginning of motion ($t = t_0$)
they retain this value)

Ex's. ① free particle

$$L = \frac{m\vec{v}^2}{2} \; : \; \underbrace{E, \; \vec{P}, \; \vec{M}}$$

all conserved, 7 quantities

however not all independent

$$E = \frac{\vec{p}^2}{2M} \quad \left(= \frac{m\vec{V}^2}{2} \right)$$

$$\underbrace{\& \; \vec{p}\cdot\vec{M} = 0} \quad \left(\text{since } \vec{M} = \vec{r} \times \vec{P} \right)$$

so only 5 are independent : as it should be –

– since there are 6 initial conditions ($\vec{r}_0$, $\vec{V}_0 = \dot{\vec{r}}_0$)

that determine solution $\vec{r} = \vec{r}_0 + \vec{V}_0(t - t_0)$ $\longrightarrow$

But the value of t_0 is arbitrary
& can not be determined by knowing $(E, \vec{M}, \vec{P})$;

on the other hand shift of $t_0 \Leftrightarrow$ shift of $\vec{r_0}$
(in dir'n of $\vec{v_0}$)

hence at most can have 5 independent
integrals of motion (and that bears true!)

Generally, if "d" degrees of freedom
have "2d" initial conditions (initial q's & $\dot{q}$'s)
but one always corresponds to a choice of t_0

→ So at most "2d − 1" independent
integrals of motion for "d" d.o.f.

Def: if more than "d" independent integrals of motion
"superintegrable" (if "d" – "integrable")

if "2d − 1" independent: "maximally superintegrable"

Ex (2): 2 particles in central $U(|\vec{r}_1 - \vec{r}_2|)$
potential: $(E, \vec{P}, \vec{M})$, again
7 integrals of motion

Combining ideas of CM frame +

+ angular momentum conservation → useful to

simplify motion in central potentials, say

(Kepler problem, H-atom ...)

— as you can tell we're in Ch. III of L&L, §13 (skipped thru "mechanical similarity" and "one-dim motion" — latter is way too simple, former will mention later!)

In an arbitrary frame two particles m_1 & m_2 $\vec{r}_1, \vec{r}_2$ interacting via $U(|\vec{r}_1 - \vec{r}_2|)$ here

$$L = \frac{1}{2} m_1 \vec{v}_1^{\,2} + \frac{1}{2} m_2 \vec{v}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|) \; ; \; \left(\vec{v}_i = \dot{\vec{r}}_i \right)$$

as we know from discussion of $\vec{R}_{CM}$

$$\vec{R}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

as there's no force that acts on the entire system, this is an inertial frame.

let's take $\vec{R}_{CM}$ as the origin of coordinates,

i.e. $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$ in this frame

and let: $\vec{r} = \vec{r}_1 - \vec{r}_2$, then

$\vec{r}_1 = \vec{r} + \vec{r}_2$ plug into $m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \Rightarrow$

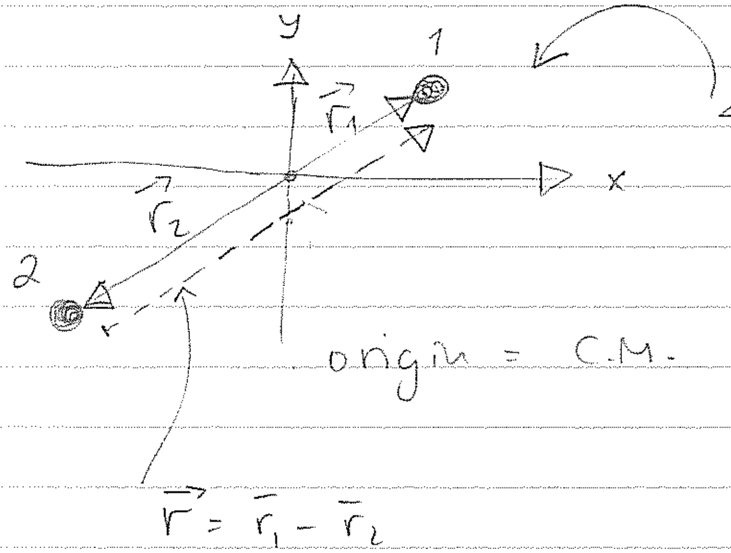
$$\Rightarrow m_1\vec{r} + m_1\vec{r}_2 + m_2\vec{r}_2 = 0 \Rightarrow \vec{r}_2 = -\frac{m_1}{m_1+m_2}\vec{r}$$

& since $\vec{r}_1 = \vec{r} + \vec{r}_2 \Rightarrow \vec{r}_1 = \vec{r} - \frac{m_1}{m_1+m_2}\vec{r} =$

$$= \frac{m_1+m_2-m_1}{m_1+m_2}\vec{r}$$

$$= \frac{m_2}{m_1+m_2}\vec{r}$$

So in c.m. we express both positions thru relative position.



Qu: as pictured, which mass is larger, m_1 or m_2?

plug $\vec{r}_1$ & $\vec{r}_2$ expressed via $\vec{r}$ into L of p. 58:

$$L = \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(-\frac{m_1}{m_1+m_2}\right)^2\dot{\vec{r}}^2 - U(|\vec{r}|)$$

$$= \frac{1}{2}(\dot{\vec{r}})^2\frac{m_1m_2^2 + m_2m_1^2}{(m_1+m_2)^2} - U(|\vec{r}|) = \frac{1}{2}(\dot{\vec{r}})^2\frac{m_1m_2}{m_1+m_2} - U(|\vec{r}|)$$

So in c.m. L looks simple —

- like that of a particle of mass

$$m_{reduced} \equiv \frac{m_1 m_2}{m_1 + m_2} \quad \left(\to m_1 \text{ if } m_2 \gg m_1 \right)$$

in a potential ("external" $U(|\vec{r}|)$:

$$L = \frac{m}{2} \dot{\vec{r}}^2 - U(|\vec{r}|)$$

(further will drop subscript "reduced" — for brevity)

(Note: so far $U(|\vec{r}|)$ vs $U(\vec{r})$ hasn't been important.)

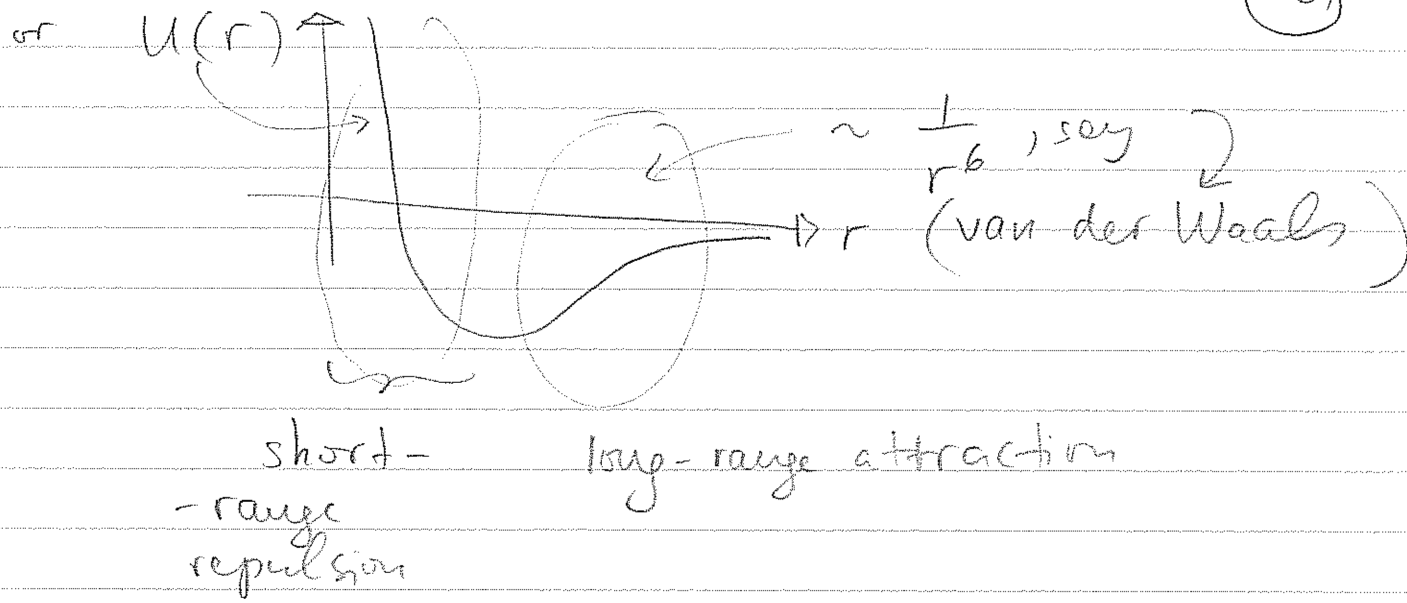
So, if we solve for $\vec{r}(t)$, can find $\bar{r}_1(t)$ & $\bar{r}_2(t)$ by using the relations of p. 59

If we have a "central" field

$$U(|\vec{r}|) = \frac{\text{const}}{r} \quad (\text{Coulomb, Newton} \cdots)$$

or

$$U(|\vec{r}|) = \frac{c e^{-r/a}}{r} \quad (\text{Yukawa -- pions -- } (N-P))$$



typical shape of interatomic inter'n – ((– "fluctuating dipoles" – ask QM))

things simplify a bit (a lot more, in fact).

Thing is L is invariant under <u>rotations</u> (in C.M., that is) $\Rightarrow$ hence $\vec{M}$ is conserved

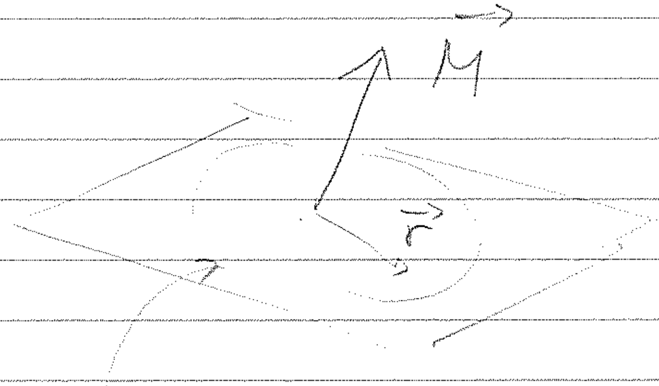
$\vec{M} = \vec{r} \times \vec{p}$ = const (means time-independent – value fixed by initial condition)

Hence, since $\vec{r} \cdot (\vec{r} \times \vec{X}) = 0$ (vector product of $\vec{r}$ w/ anything is $\perp$ to $\vec{r}$)

& since $\vec{M} = \overrightarrow{\text{const}}$ (constant vector)

we have that $\underbrace{\vec{r} \cdot \vec{M} = 0}_{\text{always true}}$ + $\underbrace{\vec{M} = \text{const}}_{\text{motion in central field}}$.

but this means that motion lies in a plane:



trajectory is PLANAR

So we choose coordinates such that $\vec{M} \parallel \hat{\vec{z}}$

then $\vec{r}$ lying in a plane means $\dot{\theta} = 0$ $\left(\theta = \frac{\pi}{2}\right)$

so we have $\dot{\vec{r}}^2 = \dot{r}^2 + r^2 \dot{\varphi}^2$

& $L = \frac{m}{2} \dot{r}^2 + \underbrace{\frac{m r^2 \dot{\varphi}^2}{2}}_{\text{this is simply related to } M_z:} - U(r)$

recall that $M_z = x \dot{y} m - y \dot{x} m$ $\quad (x(v_y m) - y(v_x m))$

$$= (r\cos\varphi)\, m (r\sin\varphi)^{\cdot} - (r\sin\varphi)(r\cos\varphi)^{\cdot}\, m =$$

$$= m r \cancel{\cos\varphi\, \dot{r} \sin\varphi} + m r \cos\varphi\, r \cos\varphi\, \dot{\varphi}$$

$$\cancel{- r\sin\varphi\, m\, \dot{r} \cos\varphi} + m r \sin\varphi\, r \sin\varphi\, \dot{\varphi}$$

$$= m r^2 \dot{\varphi} (\cos^2\varphi + \sin^2\varphi) = m r^2 \dot{\varphi} \Longrightarrow$$

hence $\quad mr^2\dot{\varphi} = M_z = \text{const.} \Rightarrow \dot{\varphi} = \dfrac{M_z}{mr^2}$

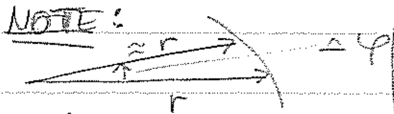
but L has $\quad mr^2\dot{\varphi}^2 = mr^2 \dfrac{M_z^2}{m^2 r^4} = \dfrac{M_z^2}{m\, r^2}$

hence, writing L ito M_z (conserved quantity)

we have

$$L = \frac{m}{2}\dot{r}^2 + \frac{1}{2}\frac{M_z^2}{m r^2} - U(r)$$

NOTE:



$\Delta(\text{area}) \approx \frac{1}{2} r^2 \Delta\varphi$

$\dfrac{\Delta(\text{area})}{\Delta\,\text{time}} =$ sectorial velocity $=$ constant

Kepler's 2nd law

a 3d problem $\Rightarrow$ a 1d problem !

What else is conserved? -- well, energy, of course. $\longrightarrow$ so let's use that $\Rightarrow$

we know $\quad E = \dfrac{1}{2}\left(m\dot{r}^2 + mr^2\dot{\varphi}^2\right) + U(r) =$

$$= \frac{1}{2} m\dot{r}^2 + \underbrace{\frac{mr^2\dot{\varphi}^2}{2}}_{\frac{M_z^2}{2mr^2} \text{ as we did above}} + U(r)$$

so $\quad E = \dfrac{1}{2} m\dot{r}^2 + \dfrac{M^2}{2mr^2} + U(r)$

(drop z from M_z —)

For this problem, we don't even need E.-L.

equations ⇒ this is because we have

2 variables that describe the trajectory ($r(t), \varphi(t)$)

– a line on the plane ⊥ $\vec{M}$

but we also have 2 integrals of motion (E, M)

⇒ so we can solve i.t.o. E & M !

("integrable system")

<u>How?</u>

well $$\frac{d\varphi}{dt} = \frac{M}{mr^2}$$ (top of p. 63)

& since $$E = \frac{m}{2}\dot{r}^2 + \frac{M^2}{2mr^2} + U(r)$$

hence
$$\dot{r}^2 = \frac{2E}{m} - \frac{2U(r)}{m} - \frac{M^2}{m^2r^2}$$
$$= \frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}$$

of course, for classically allowed motion this is always > 0

$$(*) \quad \dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}$$

hence $$dt = \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2r^2}}}$$ & $$dt = \frac{mr^2}{M}d\varphi$$

Solving (*) gives $r(t)$: for given E & M (say)

particle comes from ∞, reaches $\dot{r} = 0$, then sign

of $\sqrt{\ }$ changes (to +) & particle leaves off to ∞

But it is even more advantageous to find

equ. for path – simply replace dt by $d\varphi$ $\Rightarrow$

$$\Rightarrow \quad \frac{m r^2}{M} d\varphi = \frac{dr}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}}$$

hence $$d\varphi = \frac{M}{m} \frac{dr}{r^2} \frac{1}{\sqrt{\frac{2}{m}(E - U(r)) - \frac{M^2}{m^2 r^2}}}$$

$$d\varphi = \frac{M \frac{dr}{r^2}}{\sqrt{2m(E - U(r)) - \frac{M^2}{r^2}}}$$

so $$\underbrace{\varphi = \int \frac{M \, dr/r^2}{\sqrt{2m(E - U(r)) - M^2/r^2}}}_{\text{Equation for trajectory as } \varphi = \varphi(r)} + \text{const.}$$

Equation for trajectory as $\varphi = \varphi(r)$

| Will study integration in some important examples later |

C-3

$$L = \frac{1}{2}\mu \dot{\vec{R}}_{CM}^{\,2} + \underbrace{\frac{1}{2}\hat{\mu}\,\dot{\vec{r}}^{\,2} - U(|\vec{r}|)}_{\text{relative motion}}$$

total mass: $\mu = m_1 + m_2$

$\hat{\mu} = \dfrac{m_1 m_2}{m_1 + m_2}$ "reduced mass"

C.M.

$$\vec{P} = \mu \dot{\vec{R}}_{CM}$$

$$\vec{M}\Big|_{\text{wrt } \vec{r}=0} = \hat{\mu}\,\vec{r} \times \dot{\vec{r}}$$

$$E = \underbrace{E_{CM}}_{\frac{\mu \dot{\vec{R}}_{CM}^{\,2}}{2}} + \underbrace{E_{internal}}_{\frac{\hat{\mu}\dot{\vec{r}}^{\,2}}{2} + U(|\vec{r}|)}$$

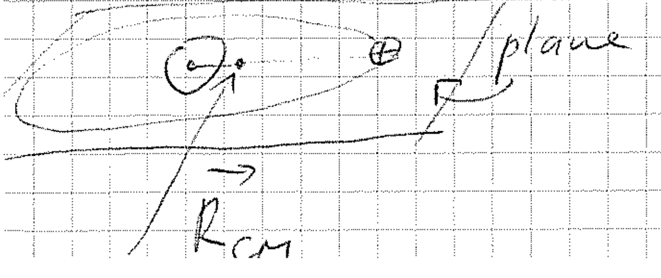
6 d.o.f.
||
d
7 (independent) integrals of motion
(II)
("superintegrable" in previous classification)

separately conserved, as $E_{CM} = \dfrac{\vec{P}^2}{2\mu}$ is conserved on its own

Since $\vec{M} = \text{const} \Rightarrow \vec{r} \times \dot{\vec{r}} = \text{const}$

both $\vec{r}$ & $\dot{\vec{r}}$ are $\perp$ to $\vec{M}$, so motion is PLANAR.

(plane is determined by $\vec{r}_0 \times \vec{v}_0$)
($\perp$ to $\vec{r}_0 \times \vec{v}_0$)



· Since now we can take

$\vec{r} = (r, \varphi)$ in plane $\perp$ $\vec{M}$

& both $E_{internal} = \frac{\mu \dot{r}^2}{2} + \frac{1}{2}\mu r^2 \dot{\varphi}^2 + U(r)$

$\underbrace{\text{\& } |\vec{M}| = \hat{\mu} r^2 \dot{\varphi} \text{ are conserved}}$

to find $r(t)$ & $\varphi(t)$

there's no need to solve any diff eqn!!

$\rightarrow \dot{\varphi} = \frac{|\vec{M}|}{\mu r^2}$, $E_{int} = \frac{\mu \dot{r}^2}{2} + \frac{|\vec{M}|^2}{2\mu r^2} + U(r)$

$\Downarrow$

like 1 dim motion !

(incl. "centrifugal barrier" potential)

recall

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{\mu}\left(E_{int} - \frac{\vec{M}^2}{2\mu r^2} - U(r)\right)$$

solution "in quadratures"

$$dt = \sqrt{\frac{\mu}{2}} \frac{dr}{\sqrt{E_{int} - \frac{\vec{M}^2}{2\mu r^2} - U(r)}}$$

trajectory

(one use $dt = \frac{\mu r^2}{|\vec{M}|} d\varphi$ to find $d\varphi = \frac{|\vec{M}|}{\mu r^2}\sqrt{\frac{\mu}{2}}\frac{dr}{\sqrt{}} \Rightarrow \varphi(r)$)

C.5

Ex 3 – gets even better if $U = \frac{\alpha}{r}$

(Kepler or H-atom)

turns out there's more symmetry !!

$E, \vec{P}, \vec{M}, \vec{A}$

$\hookrightarrow$ Runge-Lenz vector

(which allows us to skip doing $dt = \sqrt{\frac{\mu}{2}} \frac{dr}{\sqrt{E - \frac{M^2}{2mr^2} - U(r)}}$

integral!

can get orbits w/out this!)

$\vec{A} = \vec{v} \times \vec{M} - \alpha \hat{\vec{r}}$ (all $\bar{r}, \bar{v}$ refer to CM frame)

& $\underbrace{\vec{A} \cdot \vec{M} = 0}$, $\vec{A}^2 =$ (function of E & $|\vec{M}|^2$)

so really one more independent $\int$ of motion – but really all one needs to solve for trajectory of relative coordinate. (the relative coordinate problem is "maximally superintegrable": 3 dof, 5 $\int$ of motion $(\underbrace{E, \vec{M}}_{4}, \underbrace{\vec{A}}_{+1} = 5)$)

Kepler orbits via "Runge - Lenz" vector

$$\vec{M} \equiv \vec{r} \times \vec{v}\, m \quad , \quad \underline{\vec{M} = \text{const} \quad \forall\, U(r)}$$

$$(1) \quad \frac{d}{dt}\left(\vec{v} \times \vec{M}\right) = \frac{d\vec{v}}{dt} \times \vec{M} + \vec{v} \times \underbrace{\cancel{\frac{d\vec{M}}{dt}}}_{0}$$

$$(2) \quad m \frac{d\vec{v}}{dt} = -\vec{\nabla} U(r) = -\frac{dU(r)}{dr} \vec{\nabla} r$$

$$(3) \quad \vec{\nabla} r = \frac{\vec{r}}{r} \qquad \left(\vec{\nabla} \sqrt{x^2+y^2+z^2} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\right)$$

So in any radial $U(r)$ we have ((2) & (3) into (1))

$$(1) \Rightarrow \frac{d}{dt}\left(\vec{v} \times \vec{M}\right) = -\frac{1}{m}\frac{dU(r)}{dr} \vec{\nabla} r \times (\vec{r} \times \vec{v})\, m$$

$$= -\frac{U'(r)}{r} \underbrace{\vec{r} \times (\vec{r} \times \vec{v})} = *$$

for Coulomb/Kepler
$U(r) = -\frac{\alpha}{r}$, $U'(r) = \frac{\alpha}{r^2}$

use $A \times (B \times C) =$
$= B(A \cdot C) - C(A \cdot B)$

$$* = -\frac{U'(r)}{r}\left[\vec{r}\,(\vec{r}\cdot\vec{v}) - \vec{v}\, r^2\right] = -\alpha\left(\frac{\vec{r}\,(\vec{r}\cdot\vec{v})}{r^3} - \frac{\vec{v}}{r}\right)$$

---

now $\quad \hat{\vec{r}} = \frac{\vec{r}}{r}$, $\quad \frac{d}{dt}\left(\hat{\vec{r}}\right) = \frac{\dot{\vec{r}}\, r - \dot{r}\, \vec{r}}{r^2} = \frac{\vec{v}}{r} - \frac{(\vec{r}\cdot\vec{v})\,\vec{r}}{r^3}$

$$\dot{r} = \frac{d}{dt}\sqrt{\vec{r}\cdot\vec{r}} = \frac{\vec{r}\cdot\dot{\vec{r}}}{r} = \frac{\vec{r}\cdot\vec{v}}{r}$$

so we have

$$\frac{d}{dt}(\vec{v} \times \vec{M}) = \alpha \frac{d}{dt} \hat{\vec{r}}$$

Runge-Lenz vector conserved

or

$$\frac{d}{dt}\left(\vec{v} \times \vec{M} - \alpha \hat{\vec{r}}\right) = \text{const} \quad (*)$$

correct !

$$\left(\alpha = G_N M_{\odot} m \quad \text{for Kepler} \right)$$

so $\underbrace{\vec{v} \times \vec{M} = \alpha\left(\hat{\vec{r}} + \vec{e}\right)}$ is the general solution of (*)

a constant vector (constant of $\int$)

multiply by $\vec{M} \cdot$

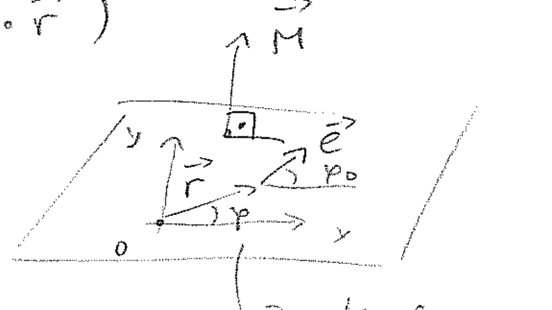
$$\Rightarrow \underset{\substack{\shortparallel \\ 0}}{\vec{M}(\vec{v} \times \vec{M})} = \alpha(\underset{\substack{\shortparallel \\ 0}}{\vec{M} \cdot \hat{\vec{r}}}) + \alpha \underbrace{\vec{M} \cdot \vec{e}}$$

$$\Rightarrow \underbrace{\vec{e} \perp \vec{M}}$$

lies in plane

multiply by $\vec{r} \cdot$

$$\underbrace{\vec{r} \cdot (\vec{v} \times \vec{M})} = \alpha\left(\hat{\vec{r}} \cdot \vec{r} + \vec{e} \cdot \vec{r}\right)$$



$$A \cdot (B \times C) = B(C \times A) = C(A \times B)$$

$$\Rightarrow \vec{M} \cdot (\vec{r} \times \vec{v}) = \frac{\vec{M}^2}{m}$$

$$\hat{\vec{r}} \cdot \vec{r} = \frac{\vec{r} \cdot \vec{r}}{r} = r \qquad \vec{e} \cdot \vec{r} = e\, r \cos(\varphi - \varphi_0)$$

$$\Rightarrow \quad \vec{M}^2 = m\alpha\left(r + e\, r\cos(\varphi-\varphi_0)\right)$$

so
$$\underbrace{r = \frac{\left(\vec{M}^2/m\alpha\right)}{1+e\cos(\varphi-\varphi_0)}}_{\text{equation of orbit.}}$$

L&L, Kepler problem or PHY254!

$\rightarrow$ go to ~~p. 87~~ $\quad p = \frac{M^2}{m\alpha}$ = "latus rectum"

$$e = \text{eccentricity} = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}$$

$$\boxed{\frac{p}{r} = 1 + e\cos\varphi \qquad \text{let } \varphi_0 = 0.}$$

extra "hidden" symmetry of Kepler problem:

$E<0$: $SO(3) \to SO(4)$

$E>0$: $SO(3) \to SO(3,1)$

in this derivation, to find e — find E of a solution — fix e

$$\frac{dr}{dt} = \frac{M^2}{m\alpha}(-)\frac{1}{(1+e\cos\varphi)^2}e(-)\sin\varphi\,\frac{d\varphi}{dt}$$

$$M = mr^2\dot\varphi \Rightarrow \frac{d\varphi}{dt} = \frac{M}{mr^2}$$

so $$\frac{dr}{dt} = \frac{M^3}{m^2\alpha}\frac{e\sin\varphi}{(1+e\cos\varphi)^2}\frac{1}{r^2} = \frac{M^3}{m^2\alpha}\frac{e\sin\varphi}{M^4}m^2\alpha^2 = e\alpha\frac{\sin\varphi}{M}$$

$\Rightarrow E = \frac{m\dot r^2}{2} + \frac{M^2}{2mr^2} - \frac{\alpha}{r}$ $\Rightarrow$ get $e(M,E)$