The general "energy conservation equation" is

$$\frac{d}{dt}L = \frac{d}{dt}\left(\vec{v}\cdot\frac{\partial L}{\partial \vec{v}}\right) + \frac{\partial L}{\partial t} \iff \frac{d}{dt}\left(\vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L\right) = -\frac{\partial L}{\partial t}$$

we called this $\mathcal{E}$ which is conserve provided $\frac{\partial L}{\partial t} = 0$

let $L = L_0 + L_1$

$$L_0 = \frac{m\vec{v}^2}{2}$$

$$L_1 = -q\phi + q\vec{v}\cdot\vec{A}$$

then we have, assume $\frac{\partial}{\partial t}L_0 = 0$:

$$\frac{d}{dt}\underbrace{\left(\vec{v}\cdot\frac{\partial L_0}{\partial \vec{v}} - L_0\right)}_{\frac{m\vec{v}^2}{2} = \mathcal{E}_0 (\equiv \mathcal{E}_{kinetic})} = -\frac{\partial L_1}{\partial t} - \frac{d}{dt}\left(\vec{v}\,\frac{\partial L_1}{\partial \vec{v}} - L_1\right)$$

$$\frac{d}{dt}(\mathcal{E}_0) = -\frac{\partial L_1}{\partial t} - \frac{d}{dt}\left(\vec{v}\,\frac{\partial L_1}{\partial \vec{v}} - L_1\right)$$

$$= +q\frac{\partial \phi}{\partial t} - q\vec{v}\cdot\frac{\partial \vec{A}}{\partial t} - \frac{d}{dt}\left(\cancel{q\vec{v}\cdot\vec{A}} + q\phi - \cancel{q\vec{v}\cdot}\right.$$

$$= \cancel{q\frac{\partial \phi}{\partial t}} - q\vec{v}\cdot\frac{\partial \vec{A}}{\partial t} - \cancel{q\frac{\partial \phi}{\partial t}} - q\vec{\nabla}\phi\cdot\vec{v}$$

$$= q\vec{v}\left(-\vec{\nabla}\phi - \partial\vec{A}/\partial t\right) = q\vec{v}\cdot\vec{E}$$

So, most proper interpretation is that

$$\frac{d}{dt}(\varepsilon_0) = q\,\vec{v}\cdot\vec{E} = q\,\frac{d\vec{r}\cdot\vec{E}}{dt}$$

$$\begin{pmatrix}\text{rate}\\ \text{of}\\ \text{change}\end{pmatrix} \text{of} \begin{pmatrix}\text{kinetic}\\ \text{energy}\\ \text{of particle}\end{pmatrix} = \begin{pmatrix}\text{work done by Lorentz force}\\ \text{in unit time}\end{pmatrix}$$

Now, for general background
$\phi = \phi(\vec{r}, t)$
$\vec{A} = \vec{A}(\vec{r}, t)$

we have

$$\frac{d}{dt}\varepsilon_0 = q\,\frac{d\vec{r}}{dt}\cdot\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right) \qquad (*)$$

It is tempting to attempt argument similar to that used in electrostatics :

from (*), if probe charge is displaced by $d\vec{r}$, the work done on it is $q\,d\vec{r}\cdot\left(-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\right)$.

Then, define its potential energy at $\vec{r}, t$ as the work needed to bring it from ∞ to $\vec{r}$, along some path, starting at $(\vec{r}_\infty, t_0) \longrightarrow (\vec{r}, t)$.

This makes perfect sense in electrostatics ...

... but let us attempt it more generally:

choose a path $\vec{r}(t)$: $\vec{r}(t_0) = \vec{r}_\infty$

$\vec{r}(t) = \vec{r}$

we want $U(\vec{r},t)$, potential energy of charge @ $\vec{r}$ at t

So integrate the r.h.s. of (*) along the path:

$$(**) \quad q \int_{t_0}^{t} dt \, \frac{d\vec{r}(t)}{dt} \left[-\vec{\nabla}\phi(\vec{r}(t), t) - \frac{\partial \vec{A}}{\partial t}(\vec{r}(t), t) \right]$$

$\vec{r}(t)$ is our path.

If we have an electrostatic field $\phi = \phi(\vec{r})$, $\vec{A} = 0$.

this (**) simplifies:

t only appears in integrand through t-dependence of $\vec{r}(t)$; so can integrate over path instead

$$q \int_{t_0}^{t} dt \, \frac{d\vec{r}}{dt} \cdot \left(-\vec{\nabla}\phi(\vec{r}(t)) \right) = q \int_{\vec{r}_\infty}^{\vec{r}} d\vec{r} \left(-\vec{\nabla}\phi \right) =$$

$$= -q\phi(\vec{r}) + q\phi(\vec{r}_\infty)$$

$\downarrow$

0.

We can also say that electrostatic field is $\phi = 0$, $\vec{A} = -\vec{E}(\vec{r})\,t$

s.t. $\vec{E}(\vec{r}) = -\dfrac{\partial \vec{A}}{\partial t}$

but it must be that $\vec{\nabla} \times \vec{E} = 0$ (if not, by Maxwell $\bar{\nabla} \times \bar{E} \sim \dfrac{\partial \vec{B}}{\partial t}$, so there's magnetic time-dep. field). If $\bar{\nabla} \times \bar{E} = 0 \Rightarrow \bar{E} = -\bar{\nabla}\phi$

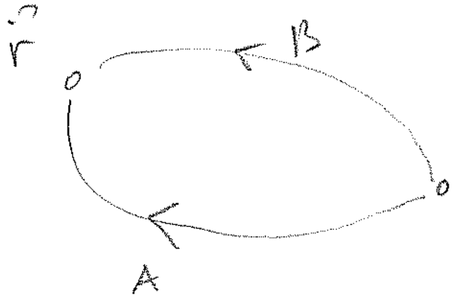
$(\bar{\nabla} \times \bar{\nabla}\phi = 0)$

then $\vec{A} = -\vec{E}(\vec{r})t = +\vec{\nabla}\phi\, t$

and we obtain above result, same as starting w/ $\phi \neq 0$

Further, if $\bar{\nabla} \times \bar{E} \neq 0$,

we have $\displaystyle\int_{r_0}^{\vec{r}} \bar{E} \cdot d\bar{r}$ depends on the path!



$$\int_{\bar{r}_0}^{\bar{r}} - \int_{\bar{r}_0}^{\bar{r}} \bar{E} \cdot d\bar{r} = \int d^2\vec{S} \cdot \bar{\nabla} \times \bar{E} \neq 0$$

(A) (B)

so $U(\bar{r})$ is path dependent.

So this def. of "potential energy" is sensible in electro-/magneto- statics only (where $\bar\nabla \times \bar E = 0$).

Back to our general conservation equ of p ①:

equ. $$\frac{d}{dt}\left(\frac{m\vec v^2}{2}\right) = q\,\bar v\cdot\bar E$$

is its only sensible interpretation ...

but we can rewrite this, of course, in different ways, if we don't split $L = L_0 + L_1$, we have

$$\frac{d}{dt}\left(\vec v\,\frac{\partial L}{\partial \bar v} - L\right) = -\frac{\partial L}{\partial t} = -\frac{\partial L_1}{\partial t} \cdot \quad \left(\frac{\partial L_0}{\partial t} = 0\right)$$

$$\Updownarrow$$

$$\frac{d}{dt}\underbrace{\left(\frac{m\vec v^2}{2} + q\phi\right)}_{} = +\,q\frac{\partial\phi}{\partial t} - q\,\vec v\cdot\frac{\partial \vec A}{\partial t}$$

this was supposed to be HW result.

this is the kinetic + pot. energy ≡ conserved quantity iff $\dot\phi = \dot{\vec A} = 0 \iff$ magneto/electro statics.

otherwise; better

$$\frac{d}{dt}\frac{m\vec v^2}{2} = q\,\bar v\cdot\bar E.$$

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