To get some IMPORTANT MATH over with

Lorentz transformations & 4-vectors

Now that we know that Einstein's relativity $\equiv$

$\equiv$ invariance of $s_{12}^2 = c^2(t_2 - t_1)^2 - (\bar{r}_2 - \bar{r}_1)^2$

$= c^2(t_2' - t_1')^2 - (\bar{r}_2' - \bar{r}_1')^2$

in any inertial frame

we need to bite the bullet and find the transformations ("Lorentz")

of $\vec{r}, t \to \vec{r}', t'$ of the coordinates of any spacetime point between O & O'.

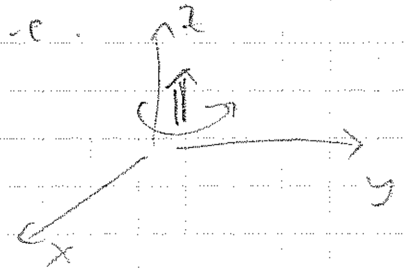
These are the generalisation of

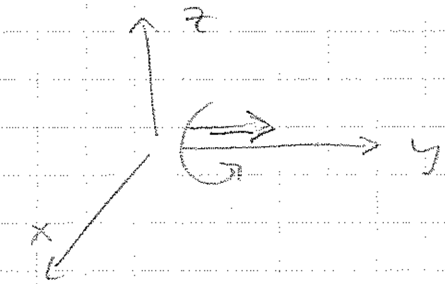
$$t' = t$$

$$\vec{r}' = \vec{r} - \vec{v}\,t \quad , \text{ Galileo's transforms,}$$

and should reduce to them in the "$c \to \infty$" limit (or $|\vec{v}|/c \to 0$ limit).

We'll argue via the close analogy of s^2 to distance $(\vec{r}_1 - \vec{r}_2)^2$ in ordinary Euclidean space. In (x, y, z)-space ($\mathbb{R}^3$) distances to the origin (or between any two points) are preserved by rotations in the x-y, y-z, z-x plane, i.e.





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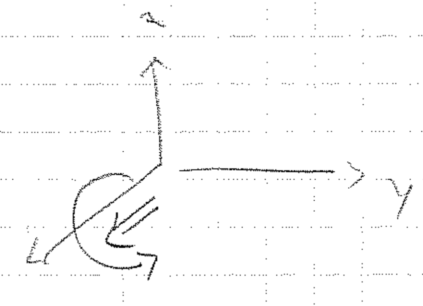


N.B.: every rotation preserving $\vec{r}^{\,2}$ can be decomposed into three rotations -- -- $SO(3)$ "group of transforms"

& what's preserved is $\underbrace{x^2+y^2+z^2}$ $\equiv$ const

spheres in $\mathbb{R}^3$ are invt under rotns.

Similarly in 4d space,

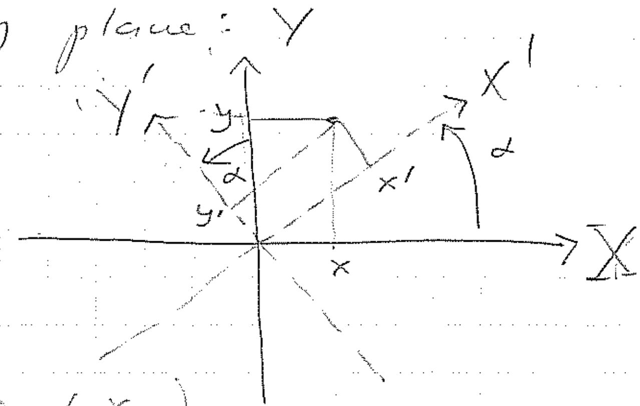
taking $(t_2, \bar{r}_2) \equiv (t, \vec{r})$; $(t_1, \bar{r}_1) \equiv (0, \vec{0})$

"rotn's" preserving $s^2 = c^2 t^2 - \vec{r}^{\,2}$ should consist of: (1) <u>ordinary rotations</u> (in x-y, y-z, z-x plane)

which preserve s^2 since clearly they preserve t & $\vec{r}^{\,2}$

(2) <u>"rotations"</u> in t-x, t-y, t-z planes

(these are called "hyperbolic" rotations, we'll see why) recall usual rotns, say in x-y plane:



$$z' = z$$

$$x' = \cos\alpha \, x + \sin\alpha \, y$$

$$y' = -\sin\alpha \, x + \cos\alpha \, y$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

—— self-read —— $\Downarrow$ ——

In the $x-t$ plane, we similarly want to 'rotate', but we want $c^2t^2 - x^2 = c^2t'^2 - x'^2$.

"Trick": let $t = i\tau/c$, $t' = i\tau'/c$ ("analytically continue")

take this as a trick to motivate eqns. on bottom of p. 30 or top of p 31

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$$\Downarrow$$

$$-\tau^2 - x^2 = -\tau'^2 - x'^2$$

$$\underbrace{\tau^2 + x^2 = \tau'^2 + x'^2}$$

BUT NOW we know what the transform is, since this allowed us to reduce s^2 to usual Euclidean distance

Transformation of $\tau \, x \to \tau' x'$ must be of the form

$$x' = \cos\xi \, x + \sin\xi \, \tau$$

$$\tau' = -\sin\xi \, x + \cos\xi \, \tau$$

But now, we know $\tau = -ict$

$\tau' = -ict'$

So we have

$$(*)\begin{cases} x' = \cos\zeta\, x + \sin\zeta(-ict) \\ -ict' = -\sin\zeta\, x + \cos\zeta(-ict) \end{cases}$$

This is nonsensical, for now, since mixes real & imaginary coordinates, but if we take $\zeta = i\psi$

$$(**)\begin{cases} \sin\zeta = \dfrac{e^{i\zeta} - e^{-i\zeta}}{2i} = \dfrac{e^{-\psi} - e^{+\psi}}{2i} = \\ \quad = -i(-)\dfrac{e^{\psi} - e^{-\psi}}{2} = i\sinh\psi \\ \cos\zeta = \dfrac{e^{i\zeta} + e^{-i\zeta}}{2} = \dfrac{e^{-\psi} + e^{+\psi}}{2} \end{cases}$$

$$= \cosh\psi$$

we'll $\underbrace{\text{obtain}}_{\text{(plugging (**))}}$ from $(*)$:

$$x' = \cosh\psi\, x + i\sinh\psi(-ict),$$

so: $\boxed{x' = \cosh\psi\, x + \sinh\psi\, ct}$, & for t':

$$-ict' = -i\sinh\psi\, x + \cosh\psi(-ict)$$

$$= (-i)\sinh\psi\, x + (-i)\cosh\psi\, ct, \text{ cancel } (-i),$$

so: $\boxed{ct' = \sinh\psi\, x + \cosh\psi\, ct}$

self-read $\Uparrow$

Hence the transforms that preserve

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

have the form

$$\left\| \begin{aligned} x' &= \cosh\psi\, x + \sinh\psi\, ct \\ ct' &= \sinh\psi\, x + \cosh\psi\, ct \end{aligned} \right\|$$

hence the name "hyperbolic rotation"

This is all good, but what is ψ?
Clearly, ψ must be related to motion of $\mathcal{O}'$ wrt $\mathcal{O}$ with some velocity V in the x direction. The origin $\mathcal{O}$ has coordinates $(t, \vec{0})$ in the $\mathcal{O}$ frame. In the $\mathcal{O}'$ frame the coordinates of the origin of $\mathcal{O}$ are

$$\left. \begin{aligned} x' &= \sinh\psi\, ct \\ ct' &= \cosh\psi\, ct \end{aligned} \right| \longrightarrow \text{hence } \underbrace{\frac{x'}{ct'}} = \tanh\psi$$

this is velocity of $\mathcal{O}$ wrt $\mathcal{O}'$ divided by c

(N.B. for those who are reading L&L, note that I am considering the same transform to that in the book — except I have the $\mathcal{O}$ & $\mathcal{O}'$ frames switched)

So, we have $\tanh \psi = V_x(\text{of } O \text{ wrt } O')/c$

$\left(\text{or } = -V_x(\text{of } O' \text{ wrt } O)/c\right)$

then, since $\cosh \psi = \dfrac{1}{\sqrt{1-\tanh^2\psi}}$

$\sinh \psi = \dfrac{\tanh \psi}{\sqrt{1-\tanh^2\psi}}$

Proof: use basics

$\sinh \psi = \dfrac{e^{\psi}-e^{-\psi}}{2}$

$\cosh \psi = \dfrac{e^{\psi}+e^{-\psi}}{2}$

$\tanh \psi = \dfrac{e^{\psi}-e^{-\psi}}{e^{\psi}+e^{-\psi}}$

$1-\tanh^2\psi = \dfrac{1}{\cosh^2\psi}$

$ch^2\psi - sh^2\psi = 1$

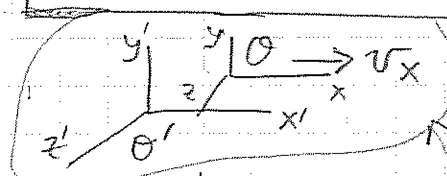
$(\cos^2\alpha + \sin^2\alpha = 1)$

we have

$$\cosh \psi = \frac{1}{\sqrt{1-\frac{V_x^2}{c^2}}}$$

$$\sinh \psi = \frac{V_x/c}{\sqrt{1-V_x^2/c^2}}$$

& transform from top of p 31 become

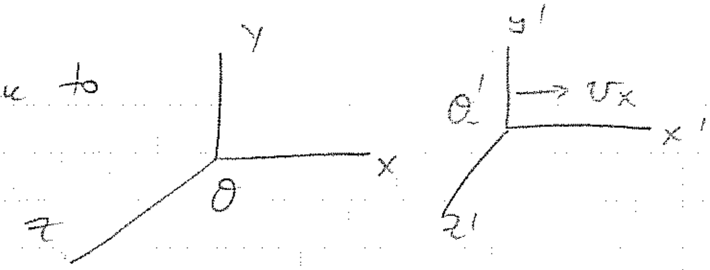


$$x' = \frac{1}{\sqrt{1-\frac{V_x^2}{c^2}}}\,x + \frac{V_x/c}{\sqrt{1-V_x^2/c^2}}\,ct \qquad y' = y$$

$$ct' = \frac{V_x/c}{\sqrt{1-\frac{V_x^2}{c^2}}}\,x + \frac{1}{\sqrt{1-V_x^2/c^2}}\,ct \qquad z' = z$$

where v_x is the velocity of O wrt O'
(unprimed) (primed)

Go back to



$$x' = \gamma\left(x - \frac{v_x}{c}\,ct\right)$$

$$ct' = \gamma\left(ct - \frac{v_x}{c}\,x\right)$$

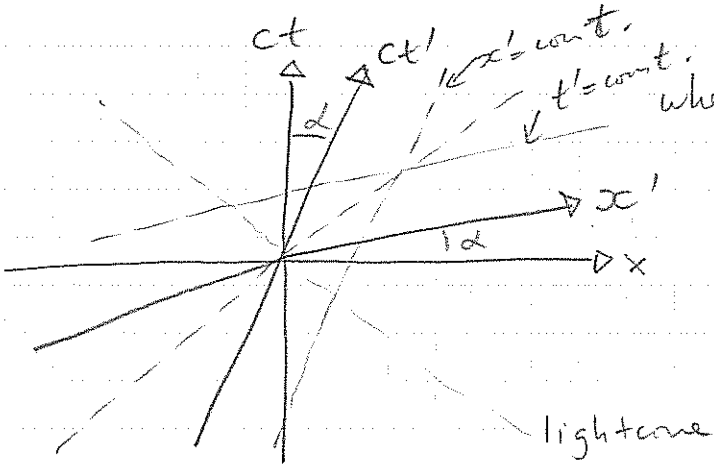
$$\gamma \equiv \frac{1}{\sqrt{1 - v_x^2/c^2}} \quad \in (1, \infty)$$

points where $x' = 0$ will give the ct' axis in x–tc plane

$x' = 0 \Rightarrow x = \frac{v_x}{c}\,ct$; Note: ct' axis $\equiv$ worldline of O' (moves w/ v_x).

points where $t' = 0$ will give x' axis in x–tc plane

$t' = 0 \Rightarrow ct = \frac{v_x}{c}\,x$



where $\tan\alpha = \frac{v_x}{c}$

Clearly, lines of constant x' & ct' are parallel to the ct' axes (constant x') and the x' axs (constant t'), respectively.
Also clear that $x' = ct'$ is the same lightcone.

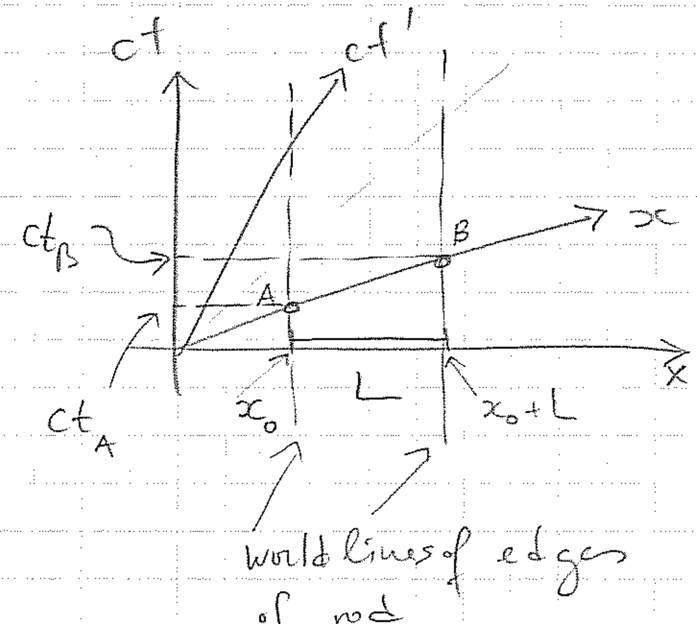
"Hyperbolic" rotation maps ct' & x' = const. lines to tilted lines on (x, ct) diagram; the angle is easy to remember by

(1) recall ct' axis $\equiv$ O' worldline (moves w/ v_x in x-ct)

(2) lightcone is invariant, so x' axis symmetric to ct' one wrt lightcone

An example of use:

let a rod be @ rest in $\mathcal{O}$



length of rod in $\mathcal{O}'$ (moving frame)

$$x'_B - x'_A = \ ?$$

(need x_B, t_B, x_A, t_A)

$A(x_0, ct_A)$

$B(x_0+L, ct_B)$

$$\frac{ct_A}{x_0} = \tan\alpha$$

$$\frac{ct_B}{x_0+L} = \tan\alpha$$

so $\quad ct_B - ct_A = L\tan\alpha$

$$x_B' - x_A' = \gamma(x_B - x_A) - \frac{v}{c}\gamma(ct_B - ct_A)$$

$$= \gamma\left(L - \frac{v}{c}L\tan\alpha\right) = \gamma L(1-\tan^2\alpha)$$

$$= \frac{1}{\sqrt{1-v^2/c^2}}\,L\left(1-\frac{v^2}{c^2}\right) = L\sqrt{1-\frac{v^2}{c^2}}$$

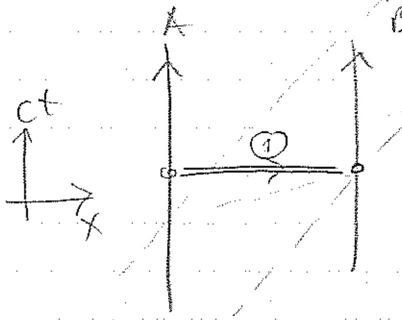
(Lorentz contracted)

even though $x_B' - x_A'$ looks "longer" in Euclidean geometry of page, it isn't ---

<u>Moral</u>: coordinates in tilted axes are not straightforward to read off diagram --- <u>But it's useful to see which events are simultaneous, or occur @ same place, in which frame</u>. (Distances in tilted x', ct' not Cartesian)

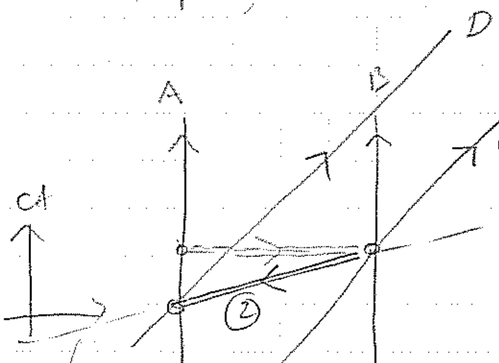
Another ex — of use of diagrams & causality ...

– suppose 'tachyons' — moving faster than light – exist and, for simplicity, allow for instantaneous transmission of information



A & B two observers

: A sends instantaneous info to B



C happens to be at same space point & same time as when B gets signal she can communicate immediately (as per her rest frame) with D

D happens to be at position of A when he gets signal. But this arrives BEFORE A even sent the signal to B in the 1st place!

this is the $t' = \text{const}$ line for C & D

<u>Moral</u> (i) A can receive answer from B <u>before</u> he sent signal ...
(— NOT SO GOOD —!)

Einstein
(ii) Relativity & instantaneous propagation of signals are incompatible
(also true for <u>faster</u> than light —)
↳ <u>see H.W.</u>

(iii) For A & B signal (2) travels backward in time, for C & D, signal (1) — — —

(iv) events which are in each others' lightcone (i.e. timelike, causally connected) always have well defined time ordering, so no problem w/ causality! ..

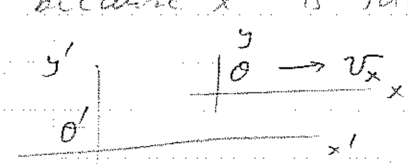
these (bottom of p. 32) are the Lorentz transforms. There are, in fact, three more (so-called "boosts") – in the y-t & z-t plane. They look just like the above, but $x \to y$
$v_x \to v_y$ (and primes).

The sign in front of terms linear wrt $\frac{v_x}{c}$ is best recalled by forgetting $\sqrt{1 - v_x^2/c^2}$ & noticing that

$$x' \approx x + v_x t + \underline{O}(v_x^2/c^2)$$

$$t' \approx t + \underline{O}\left(\frac{v_x}{c}\right)$$

So one gets Gallilean transforms and these tell us, clearly, that it is O' moving in $+x$ direction w/ $|v_x|$ (or O' moving in $-x$ w/ $|v_x|$),
(because x' is increasing w/ t for fixed x)



<u>Summary:</u> Interval $s^2 = c^2t^2 - x^2$ is preserved by $3+3=6$ kind of "rotations" $\longrightarrow$

- 3 rotations of space (xy / xz / yz)
- 3 "boosts" ("hyperbolic" rotns) in three directions $\longrightarrow$ (xt / yt / zt)

total of 6 parameters account for all the "Lorentz transformations" (discrete can be added: P & T)

rotations of space $\equiv$ SO(3) "group of transformations"

SO(3) = 3×3 orthogonal matrices w/ unit determinant

set of transforms of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ preserving $|\vec{r}|$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \hat{O} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}^T = \begin{pmatrix} x \\ y \\ z \end{pmatrix}^T \hat{O}^T$$

$\hat{O} \in SO(3)$, acting on 3 vectors

$\hat{O}^T O = \mathbb{1} \iff$ orthogonal, Recall

$$\vec{r}_2 \cdot \vec{r}_1 = (x_2 y_2 z_2) \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}^T \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} =$$

$$= \begin{pmatrix} x_2' \\ y_2' \\ z_2' \end{pmatrix}^T \underbrace{\hat{O}^T \hat{O}}_{\mathbb{1}} \begin{pmatrix} x_1' \\ y_1' \\ z_1' \end{pmatrix} = (x_2' \ y_2' \ z_2') \begin{pmatrix} x_1' \\ y_1' \\ z_1' \end{pmatrix} =$$

$$= \vec{r}_2\,' \cdot \vec{r}_1\,'$$

rotations preserve $\vec{r}_2 \cdot \vec{r}_1$

For Lorentz transforms, this generalizes very nicely --- except: (1) Lorentz transforms should involve not 3×3, but 4×4 matrices - since they act not in (x, y, z) but in (ct, x, y, z) space

(2) Lorentz transforms preserve not the $\vec{r}_1 \cdot \vec{r}_2$ inner product, but a more complicated one: $c^2 t_2 t_1 - \vec{r}_1 \cdot \vec{r}_2$

N.B.: don't get confused by this last statement ---
-- in analogy w/ rotations which preserve
$(\vec{r}_1 - \vec{r}_2)^2 = (\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)$ for all $\vec{r}_1$ & $\vec{r}_2$

we note that $(\bar{r}_1 - \bar{r}_2)^2 = \vec{r}_1 \cdot \vec{r}_1 - 2 \bar{r}_1 \cdot \bar{r}_2 - \bar{r}_2 \cdot \bar{r}_2$

but this is $(\vec{r}_1 - \vec{0})^2$ & must be invariant

this is $(\vec{r}_2 - \vec{0})^2$ & must be invariant

hence $\vec{r}_2 \cdot \vec{r}_1$ must be rotationally invariant (and it is)

→ same logic for Lorentz: if $c^2(t_2 - t_1)^2 - (\bar{r}_1 - \bar{r}_2)^2$ is

invariant, for all $(ct_1, \vec{r}_1)$ & $(ct_2, \vec{r}_2)$,
then it must be that $c^2 t_2 t_1 - \vec{r}_1 \cdot \vec{r}_2$
is also invariant.
Just like $\vec{r}$, $\vec{r}_1 \cdot \vec{r}_2$ notation is very
useful when considering nonrelativistic physics
(where space is isotropic & so laws are invariant
under rotations), it is useful to
introduce similar notation for (ct, x, y, z)

Introduce
$$\underbrace{\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}}$$
$$(x^0, x^1, x^2, x^3) \equiv \{ x^i, \ i = 0,1,2,3 \}$$

$$\{x^i\} = (ct, \vec{r})$$

If we have another 4 vector, $\{\tilde{x}^i\} = (c\tilde{t}, \vec{\tilde{r}})$,
we just argued that
$c^2 t\tilde{t} - \vec{r} \cdot \vec{\tilde{r}}$ is invariant under Lorentz
transformations. Using $\tilde{x}^i$, x^i - notation
we write this in the following form $\longrightarrow$

$$\sum_{i,j=0,1,2,3} \tilde{x}^i g_{ij} x^j = c^2 t\tilde{t} - \vec{r}\cdot\tilde{\vec{r}}$$

where $\|g_{ij}\| \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv G$

is called the

"Minkowski space metric tensor"

2-index one

$\equiv$ 4x4 matrix

$$\equiv \tilde{x}^0 \underbrace{g_{00}}_{=+1} x^0 + \tilde{x}^1 \underbrace{g_{11}}_{=-1} x^1 + \tilde{x}^2 \underbrace{g_{22}}_{=-1} x^2 + \tilde{x}^3 \underbrace{g_{33}}_{=-1} x^3$$

since all other components of $\|g_{ij}\|$ vanish

$$\equiv c\tilde{t}\,ct - \tilde{x}x - \tilde{y}y - \tilde{z}z = c^2\tilde{t}t - \tilde{\vec{r}}\cdot\vec{r}$$

(as promised)

In matrix notation

$$\sum_{ij} \tilde{x}^i g_{ij} x^j \equiv \tilde{X}^T \cdot G \cdot X$$

(a) omit limits: $ij = 0,1,2,3$ understood

(Einstein's summation rule)

(b) omit summation sign: when indices repeat, $\underline{\Sigma \text{ understood}}$

upper & lower

where $\tilde{X}^T = (\tilde{x}^0 \ \tilde{x}^1 \ \tilde{x}^2 \ \tilde{x}^3)$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

So we have $\underbrace{\tilde{X}^T \cdot G \cdot X}_{\text{so is this}} \equiv \underbrace{c^2 \tilde{t}\, t - \vec{\tilde{r}} \cdot \vec{r}}_{\text{this is Lorentz invariant}}$

On X (or $\tilde{X}$, or any 4-vector) Lorentz transforms act as a 4×4 matrix

$$X = \hat{O} X'$$

this represents 4×4 Lorentz transform, e.g.,

(for the transforms on p. 32, bottom)

$$\hat{O} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{-v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}$$

(but $\hat{O}$ can be more general – – rotations & other boosts can be involved)

Similarly $\tilde{X} = \hat{O}\,\tilde{X}'$

$$\underset{\substack{\text{4 vector}\\ \text{coordinates}\\ \text{in } O}}{X} = \underset{\text{relevant Lorentz transf.}}{\hat{O}}\ \underset{\substack{\text{4 vector}\\ \text{coordinates}\\ \text{in } O'}}{X'}$$

taking transpose, we have $\tilde{X}^T = \tilde{X}'^T \hat{O}^T$

so

$$\tilde{X}^T \cdot G \cdot X = \tilde{X}'^T \hat{O}^T \cdot G \cdot \hat{O} X'$$

|| since must be invariant ||

$$\tilde{X}'^T G X' \equiv \tilde{X}^T \hat{O}^T G \cdot \hat{O}\ X$$

must hold for <u>all</u> X & $\tilde{X}$

hence $\underbrace{G = \hat{O}^T \cdot G \cdot \hat{O}}$

<u>N.B.</u> this can be taken as a DEFINITION of Lorentz transforms ...

like $\underset{3\times 3}{\hat{O}^T}\ \underset{3\times 3}{\mathbb{1}}\ \hat{O} = \underline{\mathbb{1}}$ is a definition of $SO(3)$ matrices $\hat{O}_{3\times 3}$

3×3 matrices $\hat{O}$, s.t. $\hat{O}^T\hat{O} = \mathbb{1}$; $\det \hat{O}^T\hat{O} = 1 \Rightarrow (\det \hat{O})^2 = 1$, since $\det O^T = \det O \Rightarrow \det O = \pm 1 \rightarrow O(3)$, or if $+1$: $\underline{SO(3)}$

DEF: The Lorentz group of transformations is defined as the set of all 4×4 real matrices $\hat{O}$ such that

$$\hat{O}^T G \hat{O} = G$$

where $G = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ is the Minkowski space metric.

Similar $\det \hat{O}^T G \hat{O} = \det G = -1$

$$\det \hat{O}^T \underbrace{\det G}_{-1} \det \hat{O} = -1$$

$$(\det \hat{O})^2 = +1$$

$$\det \hat{O} = \pm 1$$

if ± 1 determinant allowed $O(1,3)$

if only $+1$ $\Rightarrow$ $SO(1,3)$

special, orthogonal; $(1,3) \longrightarrow$ one $+$, 3 $-$ elements of $G = (+1, -1, -1, -1)$

for $SO(3)$: special, orthogonal; $G^{rot} = (1,1,1)$

So: Einstein relativity $\equiv$

$\equiv$ Laws of physics invariant under $\underbrace{SO(1,3)}_{\text{Lorentz transforms (incl. space rotations)}} \times \underbrace{T_4}_{\text{translations in space time, hence } T_4}$ symmetry transforms.

together, these form "Poincare group"

[vs. nonrelativistic $SO(3) \times T_4 \times$ "Galliléo"].

very concise & precise statement of relativity principle.

We saw space time coordinates transform as 4-vectors $\{x^i\} = (ct, \vec{r})$

& inner product

$$\tilde{x}^i g_{ij} x^j$$

(Einstein summation implied)

is invariant under $SO(1,3)$

this $\tilde{x}^i g_{ij} x^j$ is very cumbersome.

Since we'll use this product so often — we'll see that we'll have to require that the Lagrangians describing relativistic particles, E & M — must all be written i.t.o. Lorentz invariants, just like the Newtonian ones must be SO(3) inv't (i.e. all dot products $\bar{r}_1 \cdot \bar{r}_2$ etc. — $\vec{A} \cdot \vec{r}$ · —),

let's absorb the g_{ij} into either $\tilde{x}$ or x:

$$\tilde{x}^i g_{ij} x^j \equiv \tilde{x}_j x^j \equiv \tilde{x}^i x_i$$

$$\tilde{x}_j = \tilde{x}^i g_{ij} = \tilde{x}^i g_{ji} \qquad (g_{ij} = g_{ji})$$

and

$$x_i = g_{ij} x^j = g_{ji} x^j$$

$$\tilde{x}_j = g_{ji} \tilde{x}^i \Rightarrow$$

$$\tilde{x}_0 = g_{00} \tilde{x}^0 + \underbrace{g_{01}}_{0} \tilde{x}^1 + \underbrace{g_{02}}_{0} \tilde{x}^2 + \underbrace{g_{03}}_{0} \tilde{x}^3 = + \tilde{x}^0$$

$$\tilde{x}_1 = g_{11} \tilde{x}^1 = - \tilde{x}^1$$

$$\tilde{x}_2 = g_{22} \tilde{x}^2 = - \tilde{x}^2$$

$$\tilde{x}_3 = g_{33} \tilde{x}^3 = - \tilde{x}^3$$

so
$$\tilde{x}_j x^j = \tilde{x}_0 x^0 + \tilde{x}_1 x^1 + \tilde{x}_2 x^2 + \tilde{x}_3 x^3$$

$$= \tilde{x}^0 x^0 - \tilde{x}^1 x^1 - \tilde{x}^2 x^2 - \tilde{x}^3 x^3$$

$$= c^2 \tilde{t} t - \vec{\tilde{r}} \cdot \vec{r}$$

So we have 4 vectors, e.g.

$$X \equiv \{x^i\} = (ct, \vec{r}), \quad X' = \hat{O} X$$

But the notion is more general —

— just like 3 vectors: $\vec{r}$ — but also $\vec{A}$ or $\vec{E}$ or $\vec{B}$

Just like all 3 vectors transform as $\vec{r}$ under $SO(3)$, all 4 vectors transform as $\{x^i\}$ (or X) under $SO(1,3)$.

The generalization of the $(\vec{r}_1 \cdot \vec{r}_2)$ "dot" product to $SO(1,3)$ is

$$\tilde{X}^T X = \tilde{x}_i x^i = \tilde{x}^i x_i = \tilde{x}^i g_{ij} x^j$$

and is a "Lorentz scalar"

↳ its value is the same in all frames

as opposed to the "Lorentz vector" $\{x^i\}$ whose components are different in different frames.

There also exist "Lorentz tensors" —

— and you already saw one of these ⟶

→ the "metric tensor"

$$\| g_{ij} \| = G = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

which has the property

$$\hat{O}^T G O = G$$

i.e. is invariant under Lorentz transformations.

A rank-k tensor is an object that has k Lorentz indices. Under Lorentz transforms, every "index gets transformed as a vector" — just like G above.

(more will follow)

simple examples

3-tensors: $I_{ij} = \sum_{a=1}^{N} m_a \left(\delta_{ij} \vec{r}_a^{\,2} - r_{ai} r_{aj} \right)$: inertia tensor of N particles

or

$I_{ij} = \int_{V_{body}} d^3\vec{r}\, \rho(\vec{r}) \left(\delta_{ij} \vec{r}^{\,2} - r_i r_j \right)$ for continuous body of mass density $\rho(\vec{r})$

Point is, kinetic energy of body wrt CM $T = \sum_{ij} \Omega_i \Omega_j I_{ij}$

is invariant under rotations of coordinate system

$T = \Omega^T \cdot I \cdot \Omega$ in matrix notation