GF. 1

Laplace Green's f-n

$$\Delta_{\vec{x}} G(\vec{x}) = -\delta^{(3)}(\vec{x})$$

$$\int d^3x\, e^{i\vec{k}\vec{x}} \Delta_{\vec{x}} G(\vec{x}) = -\int d^3x\, e^{i\vec{k}\vec{x}} \delta^{(3)}(\vec{x})$$

$$\parallel$$

$$\int d^3x \left(\Delta_{\vec{x}} e^{i\vec{k}\vec{x}}\right) G(\vec{x})$$

(+ boundary term $\to \emptyset$)

$$\parallel$$

$$-\vec{k}^2 \underbrace{\int d^3x\, e^{i\vec{k}\vec{x}} G(\vec{x})} = -1$$

$$\tilde{G}(\vec{k}) = \frac{1}{\vec{k}^2}$$

$$G(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \frac{1}{\vec{k}^2} = \frac{2\pi}{(2\pi)^3} \int_{-1}^{1} \frac{d\cos\theta \int_0^\infty dk\, k^2}{k^2} \times$$

$$\times e^{ik|x|\cos\theta} = \frac{\int_0^\infty dk}{4\pi^2} \int_{-1}^{1} d\mu\, e^{ik|\vec{x}|\mu} = \frac{1}{4\pi^2} \int_0^\infty dk \frac{e^{ik|\vec{x}|} - e^{-ik|\vec{x}|}}{i|\vec{k}||\vec{x}|}$$

$$= \frac{1}{2\pi^2} \int_0^\infty dk \frac{\sin k|\vec{x}|}{k|\vec{x}|} = \frac{1}{2\pi^2|\vec{x}|} \underbrace{\int_0^\infty d\tilde{k} \frac{\sin\tilde{k}}{\tilde{k}}}_{\frac{\pi}{2}}$$

$$= \frac{1}{4\pi|\vec{x}|} \; ; \quad \text{s.t.} \quad \Delta\varphi(x) = -4\pi\,\delta^{(3)}(x)\, q$$

$$\Rightarrow \quad \varphi(x) = \frac{q}{|\vec{x}|}$$

☒

D'Alembert ?

$$\left(\frac{\partial}{\partial x^{0\,2}} - \Delta_{\vec{x}} \right) G(x_0, \vec{x}) = -\delta^{(3)}(\vec{x})\, \delta(x^0)$$

$$\int d^3\vec{x}\, dx^0\, e^{i\vec{k}\vec{x} - ik^0x^0} \left(\partial_0^2 - \Delta_{\vec{x}}\right) G(x_0, \vec{x}) =$$

$$= -\int d^3x\, dx^0\, e^{i\vec{k}\vec{x} - ik^0x^0}\, \delta^{(3)}(\vec{x})\, \delta(x^0) = -1$$

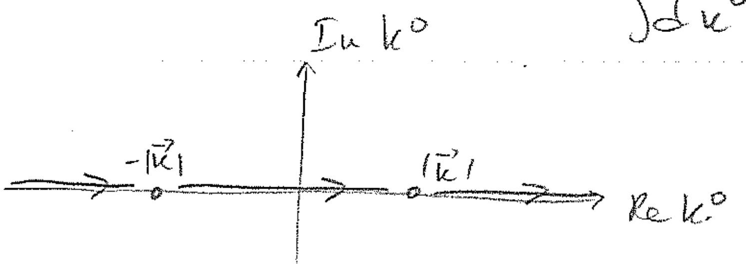
$$-1 = \left[-(k^0)^2 + \vec{k}^2\right] \underbrace{\int d^3\vec{x}\, dx^0\, e^{i\vec{k}\vec{x} - ik^0x^0} G(x_0, \vec{x})}_{\equiv\ \widetilde{G}(k^0, \vec{k})}$$

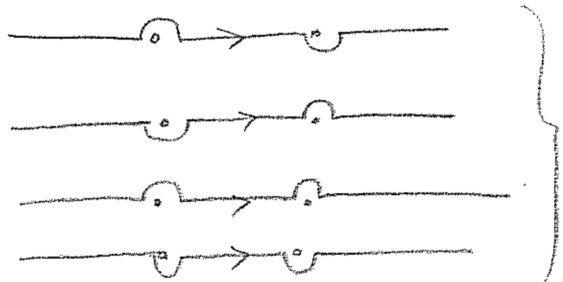
$$\widetilde{G}(k^0, \vec{k}) = \frac{1}{k^{0\,2} - \vec{k}^2} = \frac{1}{(k_0 - |\vec{k}|)(k_0 + |\vec{k}|)}$$

and
$$G(x_0, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} \underbrace{\int \frac{dk^0}{2\pi}\, e^{-i\vec{k}\vec{x} + ik^0x^0} \frac{1}{(k_0 - |\vec{k}|)(k_0 + |\vec{k}|)}}$$

<u>problem</u>: two poles on the k_0 integration contour

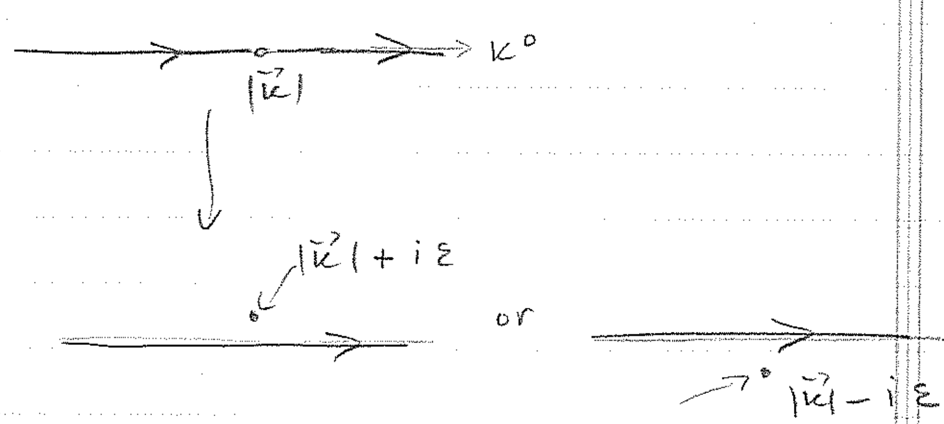
$\int dk^0$ needs definition !



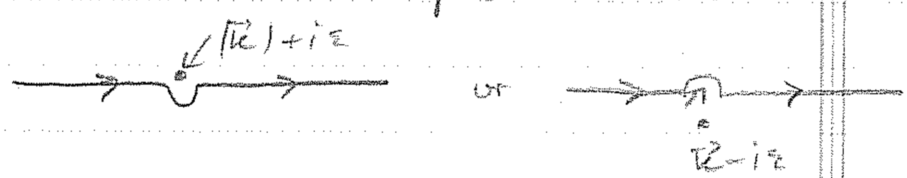


∃ 4 ways to go around pole...

equivalently, replace in each case, e.g. $(k_0 - |\vec{k}|) \longrightarrow k_0 - |\vec{k}| \pm i\varepsilon$ (& similar for $k_0 + |\vec{k}|$)



- pole shifted away from real axis
- as $\varepsilon \to 0$ ≡ push contour



<u>Which way to close?</u>: Guided by requiring $G(\vec{x}, x_0 < 0) = 0$

"retarded GF"

→ our ∫ is: $\int_{-\infty}^{\infty} dk^0 \, e^{ik^0 x^0} \frac{1}{k^{0\,2} - \vec{k}^2}$

@ $x_0 < 0$, close contour in upper half plane

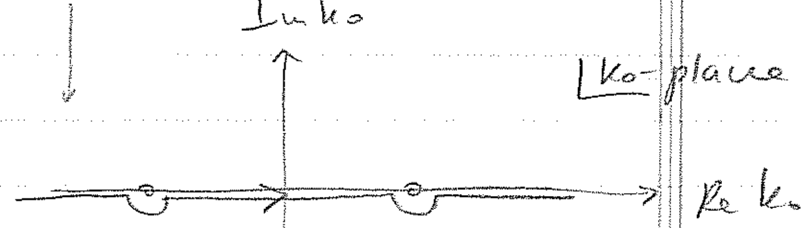
@ $x_0 < 0$ —— " —— lower half plane

↓ demand, then, that no residues inside, so $\oint = 0$.

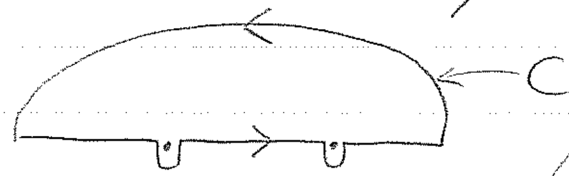
GF 4

Moral: retarded GF $\equiv$

$$\equiv \quad G(\vec{x}, x^0) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi}\, e^{ik^0x^0} \frac{1}{(k^{02}-\vec{k}^2)}$$



when $x^0 > 0$



$$\oint_C dz\, f(z) = 2\pi i \sum_{\substack{\text{res}\\ \text{inside } C}} f(z)$$

so $$G(\vec{x}, x^0>0) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\vec{x}} \frac{2\pi i}{2\pi} \sum \left(\text{Res}\, \frac{e^{ik^0x^0}}{k^{02}-\vec{k}^2}\right)\Bigg|_{\substack{k_0=|\vec{k}|\\ \& \\ k_0=-|\vec{k}|}} =$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \frac{2\pi i}{2\pi} \left(\underset{\uparrow \atop \underset{k^0=|\vec{k}|}{\text{Res}}\left(\frac{e^{ik^0x^0}}{k^{02}-\vec{k}^2}\right)}{\frac{e^{i|\vec{k}|x^0}}{2|\vec{k}|}} - \underbrace{\frac{e^{-i|\vec{k}|x^0}}{2|\vec{k}|}}_{\uparrow \atop \underset{k_0=-|\vec{k}|}{\text{Res}}\left(\frac{e^{ik^0x^0}}{k^{02}-\vec{k}^2}\right)} \right)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{-\sin(|\vec{k}|x^0)}{|\vec{k}|} e^{i\vec{k}\vec{x}} =$$

$$= \frac{-1}{4\pi^2} \int_0^\infty dk\, k \sin(kx^0) \int_{-1}^{1} d\mu\, e^{ik|\vec{x}|\mu} = -\frac{1}{4\pi^2} \int_0^\infty dk\, k \sin(kx^0) \frac{e^{ik|\vec{x}|} - e^{-ik|\vec{x}|}}{ik|\vec{x}|}$$

$$= -\frac{1}{2\pi^2|\vec{x}|}\int_0^\infty dk\, \sin(kx^0)\,\sin(k|\vec{x}|) =$$

$$= -\frac{1}{4\pi^2|\vec{x}|}\int_{-\infty}^{\infty} dk\, \underbrace{\sin(kx^0)\,\sin(k|\vec{x}|)}_{-\frac{1}{4}\left(e^{ikx^0}-e^{-ikx^0}\right)\left(e^{ikr}-e^{-ikr}\right)} \qquad r=|\vec{x}|$$

$$= +\frac{1}{16\pi^2|\vec{x}|}\int_{-\infty}^{\infty} dk\left(e^{ik(x^0+r)} - e^{ik(x^0-r)} - e^{-ik(x^0-r)} + e^{-ik(x^0+r)}\right)$$

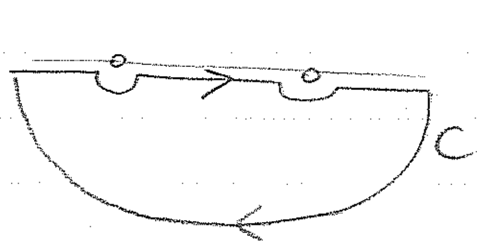
$$= \frac{1}{16\pi^2 r}\, 2\pi\left[\delta(x^0+r)-\delta(x^0-r)-\delta(x^0-r)+\delta(x^0+r)\right]$$

$$= \frac{1}{4\pi r}\left(\delta(x^0+r)-\delta(x^0-r)\right) = G(\vec{x}, x^0>0)$$

but $x^0>0$ — so $x^0+r>0 \quad \forall r$

hence $\boxed{G(\vec{x}, x^0>0) = -\frac{\delta(x^0-r)}{4\pi r}}$ — retarded

While $G(\vec{x}, x^0<0) = \int\frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\vec{x}}\underbrace{\oint_C \frac{dk^0}{2\pi}\frac{e^{ik^0x^0}}{k^{0\,2}-\vec{k}^2}}_{\equiv 0} = 0$



$\boxed{\Box G(\bar{x}, x^0) = -\delta^{(4)}(x)}$

(includes $+\,\delta^{(3)}(\vec{x})\,\delta(t)$)

$\boxed{G(\bar{x}, x^0<0) = 0}$ — retarded