Now, let's go back to the total action

$$S = \underbrace{S_{\text{matter}} + S_{\text{inter}}} + S_{\substack{EM\\ \text{field}}}$$

<u>only</u> these two contain $x^i_A(\tau)$,
dynamical D.O.F. of particles ; demanding stationarity of S wrt $\delta x^i_A(\tau)$
we got EOMs

$$\frac{d\vec{p}_A}{dt} = \left(\begin{array}{c}\text{Lorentz force on}\\ \text{A-th particle}\end{array}\right) \quad \underbrace{S_{\text{inter}} + S_{\text{EM field}}}$$

only $S_{\text{inter}} + S_{\substack{EM\\ \text{field}}}$

contain $A^i(x)$ - the dynamical D.O.F. of the EM field. Demanding (as we'll see) that the action be stationary wrt arbitrary variations of $A^i(\vec{x}, t)$, $\delta A^i(\vec{x}, t)$ (but fixed @ boundaries, as we'll describe below), we'll obtain Maxwell's equations.

Since $A^i(\vec{x}, t)$ are dynamical D.O.F. the dynamical (= mechanical) problem is formulated in the same manner as for particles.

The difference is, though, that now we have a continuum of d.o.f. (one per every $\vec{x}$).

The relevant action is

$$S[A^i] = -\frac{1}{16\pi c}\int d^4x\, F^{ij}_{(x)} F_{ij}(x) - \frac{1}{c^2}\int d^4x\, A^i_{(x)} \underbrace{j_i(x)}$$

Now we'll treat $j_i(x)$ as given "sources" (e.g. charge density & currents)

We'll demand that

$$\delta S = S[A^i + \delta A^i] - S[A^i] = \underline{O}((\delta A)^2)$$

(i.e. $S[A]$ is stationary w.r.t. variations of A^i)

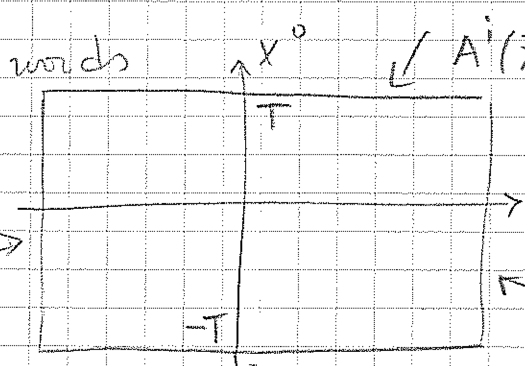
& we'll stipulate two things:

(1) at spatial infinity, i.e. far away from all charges & currents (which are the sources of A^i) we'll demand that $A^i(\bar{x}, t)\Big|_{|\bar{x}|\to\infty} \longrightarrow 0$, for all t.

Practically, this means that $\delta A^i(\bar{x}, t)\Big|_{|\vec{x}|\to\infty} = 0$.

(2) We'll consider $-T < t < T$ (T is arbitrary, really) & fix the values of $A^i(\vec{x}, -T)$ & $A^i(\bar{x}, +T)$, $\forall \vec{x}$. In other words, we fix the values of the "coordinates" $A^i(\vec{x}, t)$ ($\forall \vec{x}$) at the initial & final times, as we do for particles when we apply variational principle. This will mean that $\delta A^i(\vec{x}, \pm T) = 0$ as well, $\forall \vec{x}$.

In other words



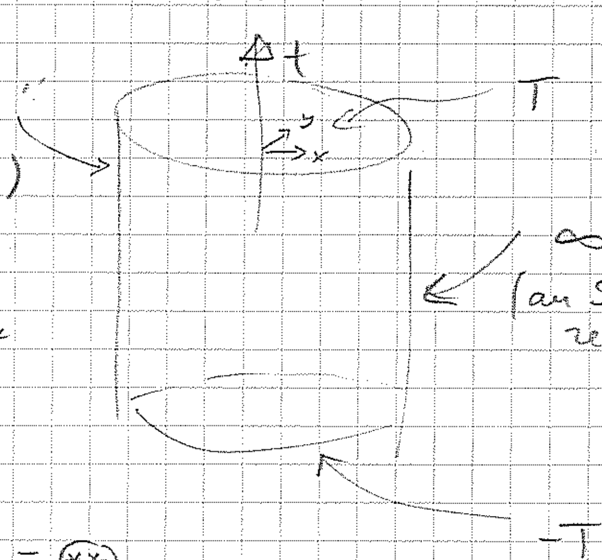
$A^i(\vec{x}, +T) =$ final values $\forall\ \bar{x}$

$A^i(\vec{\infty}, t) = 0\ ,\ t \in (-T, T)$

$A^i(\vec{\infty}, t) = 0$
$t \in (-T, T)$

$A^i(\vec{x}, -T) =$ initial values $\forall\ \bar{x}$

Note this is really a "cylinder"
$\equiv S^2_\infty \times (-T, T)$



$$\delta S[A] = -\frac{1}{16\pi c} \int d^4x \left(\delta F^{ij}(x)\, F_{ij} + \right.$$
$$\left. + F^{ij}(x)\, \delta F_{ij}\right)$$
$$- \frac{1}{c^2} \int d^4x\ \delta A^i(x)\ j_i(x) = (**)$$

(note: $\delta F^{ij} F_{ij} + F^{ij} \delta F_{ij} = \delta F_{ij}\, F^{ij} + F^{ij} \delta F_{ij} = 2 F^{ij} \delta F_{ij}$)

$$(**) = -\frac{1}{8\pi c} \int d^4x\ F^{ij}\, \delta F_{ij} - \frac{1}{c^2} \int d^4x\ \delta A^i(x)\ j_i(x) = (***)$$

$$\delta F_{ij} = \delta\left(\frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i\right) = \frac{\partial}{\partial x^i} \delta A_j - \frac{\partial}{\partial x^j} \delta A_i$$

$$(***) = -\frac{1}{8\pi c} \int d^4x \left(F^{ij} \frac{\partial\, \delta A_j}{\partial x^i} - \underbrace{F^{ij} \frac{\partial\, \delta A_i}{\partial x^j}}_{= + F^{ji} \frac{\partial \delta A_i}{\partial x^j} = F^{ij} \frac{\partial\, \delta A_j}{\partial x^i}}\right) - \frac{1}{c^2} \int d^4x\, \delta A^i j_i$$

use $F^{ji} = -F^{ji}$ — rename $i \to j\ ,\ j \to i$

$$= -\frac{1}{8\pi c}\int d^4x \left(F^{ij}\frac{\partial}{\partial x^i}\delta A_j + F^{ij}\frac{\partial}{\partial x^i}\delta A_j\right) - \frac{1}{c^2}\int d^4x\, \delta A^i j_i$$

$$= -\frac{1}{4\pi c}\int d^4x\, F^{ij}\frac{\partial}{\partial x^i}\delta A_j - \frac{1}{c^2}\int d^4x\, \delta A^i j_i =$$

$$= -\frac{1}{4\pi c}\underbrace{\int d^4x \frac{\partial}{\partial x^i}\left(F^{ij}\delta A_j\right)} + \frac{1}{4\pi c}\int d^4x\left(\frac{\partial}{\partial x^i}F^{ij}\right)\delta A_j - \frac{1}{c^2}\int d^4x\, j^i$$

$= (**)$

$$(**) \equiv \int dx^0\, d^3\vec{x}\, \frac{\partial}{\partial x^i}\left(F^{ij}\delta A_j\right) = \int d^3\vec{x}\int_{-cT}^{cT} dx^0 \frac{\partial}{\partial x^0}\left(F^{0j}\delta A_j\right)$$

$$+ \int_{-cT}^{cT} dx^0 \underbrace{\int d^3\vec{x}\, \frac{\partial}{\partial x^\alpha}\left(F^{\alpha j}\delta A_j\right)} = \int d^3\vec{x}\left(F^{0j}\delta A_j\right)(\vec{x},t)\Big|_{t=-T}^{t=T} +$$

this is the divergence of a vector

$\vec{\nabla}\cdot\vec{C}$, $C^\alpha = F^{\alpha j}\delta A_j$.

but $\delta A_j = 0$ @ $t = \pm T$ $\forall \vec{x}$.

so this term $\equiv 0$.

Use $\int_V d^3\vec{x}\, \vec{\nabla}\cdot\vec{C} = \oint_{S=\partial V} d^2\vec{S}\cdot\vec{C}$

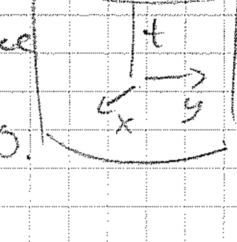
$$+ \int_{-cT}^{cT} dx^0 \oint d^2S^\alpha\, F^{\alpha j}\, \underbrace{\delta A_j(|\vec{x}|\to\infty, t)}$$

$S^2_\infty = \partial V$

$\left(S^2 \text{ @ } |\vec{x}|\to\infty \;\; \forall t\right)$

but $\delta A_j = 0$ @ S^2_∞!,

moral: boundary terms vanish, since on surface $\delta A^i = 0$.



$$\boxed{**\ *} = \delta S[A] = \int_{-cT}^{cT} dx^0 \int d^3\vec{x} \left[\delta A_j(\vec{x},x^0) \left(\frac{1}{4\pi c} \frac{\partial}{\partial x^i} F^{ij} - \frac{1}{c^2} j^j \right) \right]$$

by main theorem of variational calculus, since $\delta A_j(\vec{x}, x^0)$ is arbitrary, it must be that

$$\left\| \frac{1}{4\pi} \frac{\partial}{\partial x^i} F^{ij} = \frac{1}{c} j^j \right\| \quad (1)$$

claim: along w/ Bianchi identity

$$\epsilon^{ijkl} \frac{\partial}{\partial x^j} F_{kl} = 0 \quad \Big\| \quad (2)$$

(1) + (2) = the set of known Maxwell's eqns.

(2) $\Rightarrow$ $\boxed{\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \& \quad \vec{\nabla} \cdot \vec{B} = 0}$

(1) $\Rightarrow$ $\underline{j = 0:}$ $\frac{1}{4\pi} \frac{\partial}{\partial x^i} F^{i0} = \frac{1}{c} j^0 = \frac{1}{c} c\rho$

$i = 1,2,3$ only, since $F^{00} = 0$

$F^{i0} = \vec{E}$

so $\boxed{\vec{\nabla} \cdot \vec{E} = 4\pi \rho}$ $\longleftrightarrow$ $\boxed{\partial_i F^{i0} = \frac{4\pi}{c} j^0}$ (1^a)

$\underline{j = \alpha:}$ $\frac{4\pi}{c} j^\alpha = \frac{\partial}{\partial x^0} F^{0\alpha} + \frac{\partial}{\partial x^\beta} \underbrace{F^{\beta\alpha}}_{} = -\frac{\partial}{\partial x^0} E^\alpha + \frac{\partial}{\partial x^\beta}\left(-\epsilon^{\beta\alpha\gamma} B^\gamma\right) =$

$F^{0\alpha} = -E^\alpha$ recall: $= -\epsilon^{\beta\alpha\gamma} B^\gamma$; now use $\epsilon^{\beta\alpha\gamma} = -\epsilon^{\alpha\beta\gamma}$

$$= \frac{4\pi}{c} j^\alpha = -\frac{\partial}{c\,\partial t} E^\alpha + \epsilon^{\alpha\beta\gamma} \frac{\partial}{\partial x^\beta} B^\gamma$$

$$-\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} + \vec{\nabla}\times\vec{B}$$

or

$$\boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}} \iff \boxed{\partial_i F^{i\alpha} = \frac{4\pi}{c} j^\alpha}$$

1^b

4-vector form		3-vector form
$\epsilon^{ijkl}\,\partial_j F_{kl} = 0$	$\iff$	$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
		$\vec{\nabla}\cdot\vec{B} = 0$
$\partial_i F^{ij} = \frac{4\pi}{c} j^j$	$\iff$	$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$
		$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

Comparison of these eqns to Griffiths (or whatever you're used to) defines the system of units. Note the pleasant fact that no ε_0, μ_0 etc — unphysical stuff — appear here, but only $\underline{c}$ — which, we'll see soon, is indeed the 'speed of light'

($\equiv$ EM waves)

Let's see how 'c' comes about to be the speed of light... (we'll, afterwards, come back to many more issues we need to understand, namely the energy, momentum, angular momentum of the EM field; the energy of a system of charges, etc.

It is convenient to study $\underbrace{A^i \quad (\vec{A}, \varphi)}_{\text{equations in terms of}}$.

Remember $\quad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$

$\vec{B} = \vec{\nabla} \times \vec{A}$.

Consider the Maxwell equations in vacuum (or, if you wish, outside of sources); the ones which do not follow from def of $\vec{E}$ & $\vec{B}$ i.t.o. $(\phi, \vec{A})$ are:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \longrightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\frac{1}{c}\vec{\nabla}\frac{\partial \phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$$

$$\vec{\nabla} \cdot \vec{E} = 0 \longrightarrow -\vec{\nabla} \cdot \vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = 0 \qquad (109.1)$$

Remember also that $(\phi, \vec{A}) \sim (\phi', \vec{A}')$

physically equivalent, i.e. same $\vec{E}, \vec{B}$

where $\phi = \phi' + \frac{\partial \chi}{\partial (ct)}$, $\vec{A} = \vec{A}' - \vec{\nabla}\chi$, where $\chi(\bar{x}, t)$ is arbitrary.

For any $\{\phi(\bar{x}, t)\}$, there exists an $\chi(\bar{x}, t)$ s.t. $\{\phi'(\bar{x}, t)\} = 0$

field configuration, i.e. the values $\forall \bar{x}, t$

Thus, one can eliminate ϕ from the equations!

Is that really so? – Since $\phi = \phi' + \frac{1}{c}\frac{\partial \chi}{\partial t}$,

all we need to show is that $\exists\, \chi$

$$((*))\ \frac{\partial \chi}{\partial t} = c\,\phi(\bar{x},t) \quad , \quad \forall\, \phi \quad (\text{i.e. } \chi \text{ is such that } \phi' = 0)$$

– but χ clearly exists. $\chi(\bar{x},t) = c\int_{-\infty}^{t} dt'\,\phi(\bar{x},t')$

satisfies ((*)). Then, working w/ $\vec{A}'$ & $\phi' = 0$,

we have from (109.1):

$$\left.\begin{aligned} &\vec{\nabla}\times(\vec{\nabla}\times\vec{A}') + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}' = 0 \\ &\qquad\qquad \| \\ &-(\vec{\nabla}^2)\vec{A}' + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}') + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}' = 0 \\ \& \quad & \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}') = 0 \end{aligned}\right\| \quad (110.1)$$

Now, we can perform another gauge transformation $\tilde{\chi}(\bar{x},t)$

$(\phi' = 0, \vec{A}') \longrightarrow (\phi'' = 0, \vec{A}'')$, which will

not lead to $\phi'' \neq 0$ if the corresponding $\tilde{\chi}(\bar{x},t)$

is t-independent (since $\phi' = \phi'' + \frac{1}{c}\cancel{\frac{\partial \tilde{\chi}}{\partial t}} \to 0$).

$$0 = 0$$

Furthermore, $\vec{A}' = \vec{A}'' - \vec{\nabla}\tilde{\chi}(\vec{x})$. We can choose

$\tilde{\chi}(\vec{x}')$ to impose a further condition on $\vec{A}''$, namely

that $\vec{\nabla}\cdot\vec{A}'' = 0$. Note that by second eqn (110.1),

$\vec{\nabla}\cdot\vec{A}'$ is t-independent, hence

$$\vec{\nabla}\cdot\vec{A}' = \underbrace{\vec{\nabla}\cdot\vec{A}''}_{0} - \vec{\nabla}^2\tilde{\chi}(\vec{x}) \Leftarrow \left(\vec{\nabla}\cdot\left[\vec{A}' - \vec{A}'' + \vec{\nabla}\tilde{\chi}\right] = 0 \right.$$

0 - our desired condition

t-independent by (109.1) - 2nd eqn

can be always solved:

$$\vec{\nabla}^2\chi(\vec{x}) = -\vec{\nabla}\cdot\vec{A}'(\vec{x})$$

(this is Laplace's eqn w/ "charge density" $\sim \vec{\nabla}\cdot\vec{A}'$; a "potential" $\tilde{\chi}(\vec{x})$ always exists) ! physicist's proof

Moral: any solution of the vacuum M. eqns (109.1) can be cast into "Coulomb gauge", i.e. can be taken to obey $\phi = 0$, $\vec{\nabla}\cdot\vec{A} = 0$!

(one can also write solution via Green's functions)

So, in terms of $(\phi'', \vec{A}'')$, $\vec{\nabla}\cdot\vec{A}'' = 0$, we have
$\phi'' = 0$

$$\text{from (109.1)} \Rightarrow \begin{cases} -(\vec{\nabla}^2)\vec{A} + \dfrac{1}{c^2}\dfrac{\partial}{\partial t^2}\vec{A} = 0 \\ \vec{\nabla}\cdot\vec{A} = 0, \quad \phi = 0 \end{cases}$$

(where we dropped the $\bar{A}''$'s, ϕ'''s.)

Maxwell's eqns in vacuum in "Coulomb gauge" $\phi = 0$, $\bar{\nabla}\cdot\bar{A} = 0$

$$\vec{\nabla}^2 \equiv \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \text{3d Laplacian}$$

$\vec{A}$ obeys:
$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\vec{A} - \Delta\vec{A} = 0 \\ \vec{\nabla}\cdot\vec{A} = 0 \end{cases}$$

<u>Claim</u>: this is the <u>WAVE EQUATION</u> for EM waves. EM waves propagate <u>w/ speed: c</u> and are <u>transverse</u>. (to be substantiated).

<u>Note</u>: "Coulomb gauge" condition $\phi = 0$, $\vec{\nabla}\cdot\vec{A} = 0$ is not Lorentz invariant, e.g. if $\phi = 0$ in one frame, $\phi \neq 0$ in another frame. Thus, if one uses it, one must be mindful of which frame it's valid in (but there's nothing wrong with it, so long as one remembers).

Another useful "gauge condition" is the relativistically invariant "Lorentz gauge". To see how this works, start from
$$\partial_i F^{ij} = 0 \quad \text{(same set of M eqns in vacuum)}$$
or
$$\partial^i F_{ij} = 0 = \partial^i(\partial_i A_j - \partial_j A_i) = \partial^i\partial_i A_j - \partial^i\partial_j A_i =$$
$$= \partial^i\partial_i A_j - \partial_j(\partial^i A_i) = \underbrace{\partial^i\partial_i A_j - \partial_j(\partial_i A^i) = 0}_{\text{Maxwell's eqn in vacuum.}}$$

Now,

given $A^i(x)$, we can always find χ,

$$A^i = A^{i'} + \partial^i \chi \quad \text{s.t.} \quad \partial_i A^{i'} = 0.$$

thus, one must show that χ exist, s.t.

$$\partial_i A^i = \underbrace{\partial_i A^{i'}}_{0} + \partial_i \partial^i \chi \quad \text{or}$$

$$\underbrace{\partial_i \partial^i}\, \chi = \partial_i A^i(x)$$

$$\partial_0^2 - \partial_\alpha^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \; : \; \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) \chi(\vec{x},t) = \partial_i A^i(\vec{x},t)$$

We'll take it for granted that such χ exists, therefore we can always take $\partial_i A^i = 0$ (a Lorentz scalar condition)

physicist proof: χ is the wave created by an arbitrary source $(\partial_i A^i)(\vec{x},t)$; it exists ...

mathematical (equivalent): write solution for χ using Green's functions ...

hence equ. on bottom of p 112 gives
$$\begin{cases} \partial_i \partial^i A_j = 0 \\ \partial^j A_j = 0 \end{cases}$$

but this is, using $\partial_i \partial^i = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$

$$\begin{cases} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) A_j = 0 \\ + \quad \partial^j A_j = 0 \iff \frac{1}{c}\frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0. \end{cases}$$

Maxwell's equs in vacuum in "Lorentz gauge"