We showed that Maxwell's eqns in vacuum lead to either:

"Coulomb gauge"

$$A^0 = 0$$

$$\vec{\nabla}\cdot\vec{A} = 0$$

$$\left(\frac{1}{c^2}\frac{\partial}{\partial t^2} - \Delta\right)\vec{A} = 0$$

"Lorentz gauge"

$$\partial_i A^i = 0 \quad (*)$$

$$\left(\frac{1}{c^2}\frac{\partial}{\partial t^2} - \Delta\right) A^i = 0$$

(*) is further invariant under $A_i \to A_i + \partial_i \chi$ w/ $\partial^i \partial_i \chi = 0$, i.e. $\left(\frac{1}{c^2}\frac{\partial}{\partial t^2} - \Delta\right)\chi = 0$, so gauge freedom is not completely eliminated.

In each case we have an eqn:

$$\left(\frac{1}{c^2}\partial_{t^2} - \Delta\right) f = 0$$

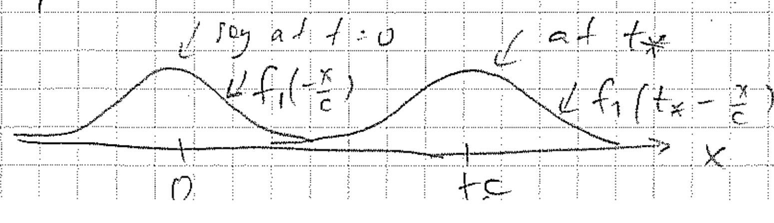
$\downarrow$ in 1dim $\Delta = \frac{\partial^2}{\partial x^2}$

so $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f(t,x) = 0 \to$ this is called generally, a "wave equation" and the general solution is

$$f(t,x) = f_1\left(t - \frac{x}{c}\right) + f_2\left(t + \frac{x}{c}\right), \text{ where}$$

f_1 & f_2 are arbitrary functions of their arguments

$f_1\left(t - \frac{x}{c}\right)$ means f_1 is constant at $x = tc$



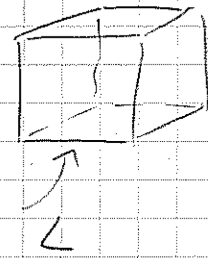
f_1 : wave moving in $+x$ w/ c

f_2 : wave moving in $-x$ w/ c

Clearly 'c' is the speed of the waves.

From the wave equation (in either gauge), we can learn about the properties of $\vec{E}$ & $\vec{B}$ (the physical, gauge invariant, content of an EM wave).

Let's work in Coulomb gauge, for definiteness.
Then we have $\vec{A}(\vec{x},t)$ obeying $\vec{\nabla}\cdot\vec{A}(\vec{x},t) = 0$
& the wave equation $\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\vec{A} = 0$.

Imagine, for a moment, that $\vec{x} \in$  $V = L^3$,

and that $\vec{A}(\vec{x} + \vec{e}_i L, t) = \vec{A}(\vec{x}, t)$.
(We'll take $L \to \infty$ in the end of the day, for now this is a trick to introduce Fourier transform.)

=== LITTLE MATH INTERLUDE ===

For a f-n of one variable $f(x+L) = f(x)$ we can always write
$$f(x) = \sum_{n=-\infty}^{+\infty} e^{i\frac{2\pi n}{L}x}\,\tilde{f}_n \,,$$ where,
if $f(x) = f^*(x)$ (f is real) we must have $\tilde{f}_n = \tilde{f}_{-n}^*$.

$\tilde{f}_n$ is called the "Fourier component" of $f(x)$; let's call

$k_n = \frac{2\pi n}{L}$ – the "wave vector", for now just a name.

So we have $f(x) = \sum_{n=-\infty}^{\infty} \tilde{f}_{k_n}\, e^{ik_n x}$ ($\tilde{f}_{k_n}$ = FT of $f(x)$

the inverse relation is

$$\tilde{f}_{k_n} = \frac{1}{L}\int_{-L/2}^{L/2} dx\, f(x)\, e^{-ik_n x}$$

to see this, put $f(x)$ into inv. FT:

$$f(x) = \sum_m \tilde{f}_{k_m} e^{ik_m x}$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} dx \sum_m \tilde{f}_{k_m} e^{i(k_m - k_n)x} =$$

$$= \sum_m \underbrace{\frac{1}{L}\int_{-L/2}^{L/2} dx\, e^{i\frac{2\pi x}{L}(m-n)}}_{L\,\delta_{mn}} \tilde{f}_{k_m}$$

$$= \sum_m \delta_{mn} \tilde{f}_{k_m} = \tilde{f}_n \quad \square$$

get back LHS. ✓

It is conventional to absorb L into $\tilde{f}_{k_n}$

let $\tilde{f}_{k_n} L \equiv \tilde{f}(k_n)$

then $$f(x) = \frac{1}{L}\sum_n \tilde{f}(k_n) e^{ik_n x}, \quad k_n = \frac{2\pi n}{L}$$

$$\tilde{f}(k_n) = \int_{-L/2}^{L/2} dx\, f(x)\, e^{-ik_n x}$$

Finally, one takes $L \to \infty$ like this: $k_n = \frac{2\pi n}{L}$, so

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when n changes by $\Delta n = 1$ we have

$$\Delta k_n = \frac{2\pi}{L}\Delta n \,, \quad \text{so} \quad \Delta n = \frac{L}{2\pi}\Delta k_n \quad \left(\text{notice } \begin{matrix} L \to \infty \\ \Delta k_n \to 0 \end{matrix}\right)$$

hence $$f(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \Delta n \, \tilde{f}(k_n) \, e^{ik_n x} =$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta k_n \, \tilde{f}(k_n) \, e^{ik_n x} \xrightarrow[\Delta k_n \to 0]{L \to \infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, \tilde{f}(k) e^{ikx}$$

& we have $$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) \, e^{ikx}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \, f(x) \, e^{-ikx}$$, where now

one checks that $$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tilde{f}(p) \, e^{i(p-k)x} =$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \tilde{f}(p) \underbrace{\int_{-\infty}^{\infty} dx \, e^{i(p-k)x}}_{= 2\pi\,\delta(p-k)} = \int_{-\infty}^{\infty} dp \, \delta(p-k) \, \tilde{f}(p) = \tilde{f}(k)$$

$$\left(\text{before this was } \int_{-L/2}^{L/2} dx \, e^{ix\frac{2\pi}{L}(n-m)} = L\,\delta_{nm} \right)$$

END OF INTERLUDE

We now do this w/ $\vec{A}(\vec{x}, t)$ periodic in x, y, z $\longrightarrow$

$\longrightarrow$ and take $L \to \infty$

right away

$$\vec{A}(\vec{x},t) = \iiint_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3}\, \vec{\tilde{A}}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\vec{\tilde{A}}(\vec{k},t) = \iiint_{-\infty}^{\infty} d^3x\, \vec{A}(\vec{x},t)\, e^{-i\vec{k}\cdot\vec{x}}$$

(where now $\int d^3x\, e^{i(\vec{k}-\vec{p})\cdot\vec{x}} = (2\pi)^3\, \delta^{(3)}(\vec{k}-\vec{p})$

$\Rightarrow$ we now have the FT, $\vec{\tilde{A}}(\vec{k},t)$ of $\vec{A}(x)$.

Since $\vec{A}(\vec{x})$ is real $\Rightarrow$ $\vec{\tilde{A}}(\vec{k},t)^* = \vec{\tilde{A}}(-\vec{k},t)$.

But our $\vec{A}(\vec{x},t)$ also obeys $\vec{\nabla}\cdot\vec{A}(\vec{x},t) = 0$.

In $\vec{k}$-space.

$$\vec{\nabla}\cdot\vec{A}(\vec{x},t) = 0 = i\int \frac{d^3k}{(2\pi)^3}\, \vec{k}\cdot\vec{\tilde{A}}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

because this is true $\forall\, \vec{x}$, it must be that

$$\vec{k}\cdot\vec{\tilde{A}}(\vec{k},t) = 0.$$

Finally, $\vec{A}(\vec{x},t)$ obeys $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla}\right)\vec{A}(\vec{x},t) = 0$

wave equation

what does this mean for FT ?

$$0 = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \vec{A}(\vec{x}, t) =$$

$$= \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \int \frac{d^3k}{(2\pi)^3} \vec{\tilde{A}}(\vec{k}, t)\, e^{i \vec{k} \cdot \vec{x}} =$$

$$= \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{c^2} \ddot{\vec{\tilde{A}}}(\vec{k}, t)\, e^{i\vec{k}\cdot\vec{x}} - \underbrace{\frac{\partial}{\partial \vec{x}} \cdot \frac{\partial}{\partial \vec{x}} e^{i\vec{k}\cdot\vec{x}}}_{+\vec{k}^2 e^{i\vec{k}\cdot\vec{x}}} \vec{\tilde{A}}(\vec{k}, t) \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \times$$

$$\times \underbrace{\left(\frac{1}{c^2} \ddot{\vec{\tilde{A}}}(\vec{k}, t) + \vec{k}^2 \vec{\tilde{A}}(\vec{k}, t) \right)}_{} = 0$$

$$\Rightarrow \left\| \begin{array}{l} \ddot{\vec{\tilde{A}}}(\vec{k}, t) = -c^2 \vec{k}^2 \vec{\tilde{A}}(\vec{k}, t) \\ \vec{k} \cdot \vec{\tilde{A}}(\vec{k}, t) = 0 \qquad \& \quad \vec{\tilde{A}}(-\vec{k}, t) = \vec{\tilde{A}}^{*}(\vec{k}, t) \end{array} \right.$$

the Fourier modes of the $\vec{A}(\vec{x}, t)$, one per each $\vec{k}$, obey a <u>simple harmonic oscillator</u> equation.

the frequency of the oscillator is

$$\omega_{\vec{k}} = c|\vec{k}| \, . \qquad \ddot{\vec{\tilde{A}}}(\vec{k}, t) = -\omega_k^2 \vec{\tilde{A}}(\vec{k}, t) \, .$$

$\longrightarrow$ but we know how to solve this !!

But before we continue, take stock:

$$\vec{A}(\bar{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\bar{k}\bar{x}}\, \vec{\tilde{A}}(\vec{k},t) \quad (120.1)$$

$$\vec{k}\cdot\vec{\tilde{A}}(\vec{k},t) = 0 \quad (120.2)$$

$$\vec{\tilde{A}}(\vec{k},t) = \vec{\tilde{A}}^*(-\vec{k},t) \quad (120.3)$$

$$\& \quad \ddot{\vec{\tilde{A}}}(\vec{k},t) = -\omega_k^2\, \vec{\tilde{A}}(\vec{k},t), \quad \omega_k = c|\vec{k}| \quad (120.4)$$

to deal w/ this condition, proceed this way:

(120.1)

$$\vec{A}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\bar{k}\cdot\bar{x}}\, \frac{1}{2}\Big(\vec{\tilde{A}}(\vec{k},t) + \underbrace{\vec{\tilde{A}}(\vec{k},t)}_{\vec{\tilde{A}}^*(-\vec{k},t)}\Big) =$$

by (120.3)

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\Big(\vec{\tilde{A}}(\vec{k},t) + \vec{\tilde{A}}^*(-\vec{k},t)\Big) \quad (120.5)$$

(120.3) has general solution $\vec{\tilde{A}}(\vec{k},t) = e^{i\omega_k t}\,\vec{a}_+(\vec{k}) + e^{-i\omega_k t}\,\vec{a}_-(\vec{k})$

(hence $\vec{\tilde{A}}^*(-\vec{k},t) = e^{-i\omega_k t}\,\vec{a}^*_+(-\vec{k}) + e^{i\omega_k t}\,\vec{a}^*_-(-\vec{k})$)

into (120.5)

$$\vec{A}(\bar{x},t) = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\, e^{i\bar{k}\bar{x}}\Big(e^{i\omega_k t}\big(\vec{a}_+(\vec{k}) + \vec{a}^*_-(-\vec{k})\big) + e^{-i\omega_k t}\big(\vec{a}_-(\vec{k}) + \vec{a}^*_+(-\vec{k})\big)\Big)$$

(120.6)

$(+\text{recall } \vec{k}\cdot\vec{a}_\pm(\vec{k}) = 0.)$

$$\vec{A}(\vec{x},t) = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\Big[e^{i\vec{k}\cdot\vec{x}+i\omega_k t}\underbrace{\left(\vec{a}_+(\vec{k})+\vec{a}^{\,*}_-(-\vec{k})\right)}_{\equiv\ \vec{\beta}^{\,*}(-\vec{k})\,2}$$

$$+\,e^{i\vec{k}\vec{x}-i\omega_k t}\underbrace{\left(\vec{a}^{\,*}_+(-\vec{k})+\vec{a}_-(\vec{k})\right)}_{\equiv\ \vec{\beta}(\vec{k})\,2}\Big]$$

$$\vec{A}(\vec{x},t) = \int\frac{d^3k}{(2\pi)^3}\Big[e^{i\vec{k}\vec{x}+i\omega_k t}\,\vec{\beta}^{\,*}(-\vec{k}) + e^{i\vec{k}\vec{x}-i\omega_k t}\,\vec{\beta}(\vec{k})\Big] \quad + \quad \vec{k}\cdot\vec{\beta}(\vec{k}) = \qquad (121.1)$$

N.B. This gives a manifestly real expression for $\vec{A}(\vec{x},t)$

e.g. c.c. 1st term, get $\int e^{-i\vec{k}\vec{x}-i\omega_k t}\,\vec{\beta}(-\vec{k})\,d^3\vec{k}$,

but then change $\vec{k} = -\vec{k}'$ $\qquad \iiint_{-\infty}^{\infty} d^3k = -\iiint_{+\infty}^{-\infty} d^3k' =$

and get

$$\int d^3k'\, e^{ik'x-i\omega_{k'}t}\,\vec{\beta}(\vec{k}') \qquad = +\iiint_{-\infty}^{\infty} d^3k'$$

which, upon dropping primes, gives 2nd term ∎

(121.1) says: <u>Most general solution of vacuum M. eqns is a superposition of plane monochromatic waves</u> $\longrightarrow$ so let's study their properties!

$\longrightarrow$

let's take $\vec{\rho}(\vec{k}) = \vec{\beta}\,\delta^{(3)}(\vec{k}-\vec{p})(2\pi)^3$

with real $\vec{\beta}$

hence $\vec{\rho}^{\,*}(-\vec{k}) = \vec{\beta}\,\delta^{(3)}(-\vec{k}-\vec{p})(2\pi)^3$; plug

into (121.1)

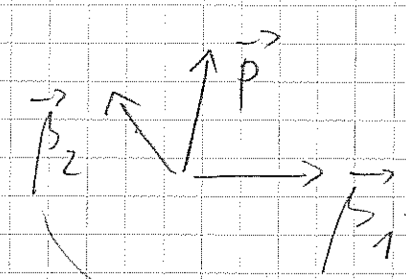
$$\vec{A}(\vec{x},t) = \left(e^{-i\vec{p}\vec{x}+i\omega_p t} + e^{i\vec{p}\vec{x}-i\omega_p t}\right)\vec{\beta}$$

$$\begin{cases} \vec{A}(\vec{x},t) = \vec{\beta}\cos(\omega_p t - \vec{p}\vec{x}) \\ \vec{p}\cdot\vec{\beta} = 0 \qquad (\vec{\nabla}\cdot\vec{A} = 0) \end{cases}$$

Let $\vec{p} = (p,0,0)$, $\vec{p}\cdot\vec{\beta} = 0 \Rightarrow \beta_x = 0$, $\vec{\beta} = (0,\beta_y,\beta_z)$

Hence, there are <u>two linearly independent choices</u> for $\vec{\beta}$

$\forall\,\vec{p}$:



these will be called "<u>polarizations</u>"

Now, $\vec{E} = -\dfrac{\partial}{\partial\vec{x}}\phi - \dfrac{\partial}{c\,\partial t}\vec{A}$ & $\vec{B} = \vec{\nabla}\times\vec{A}$

(0, we're in Coulomb gauge $\phi = 0$, $\vec{\nabla}\cdot\vec{A} = 0$)

$$\vec{E}(x,t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\beta}\cos(p(ct-x)) = \vec{\beta}\,p\sin(p(ct-x)).$$

(123)

to find $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\left(\cos(\omega_p t - \vec{p}\cdot\vec{x})\right)\times\vec{\beta} =$

$$= -\sin(\omega_p t - \vec{p}\cdot\vec{x})\,(-\vec{p}\times\vec{\beta})$$

$$\vec{B} = \vec{p}\times\vec{\beta}\,\sin(\omega_p t - \vec{p}\cdot\vec{x})$$

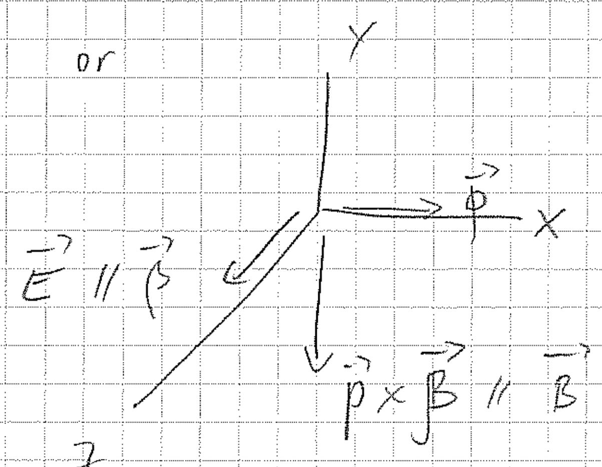
$$\vec{B} = \vec{p}\times\vec{\beta}\,\sin(p(ct - x))$$

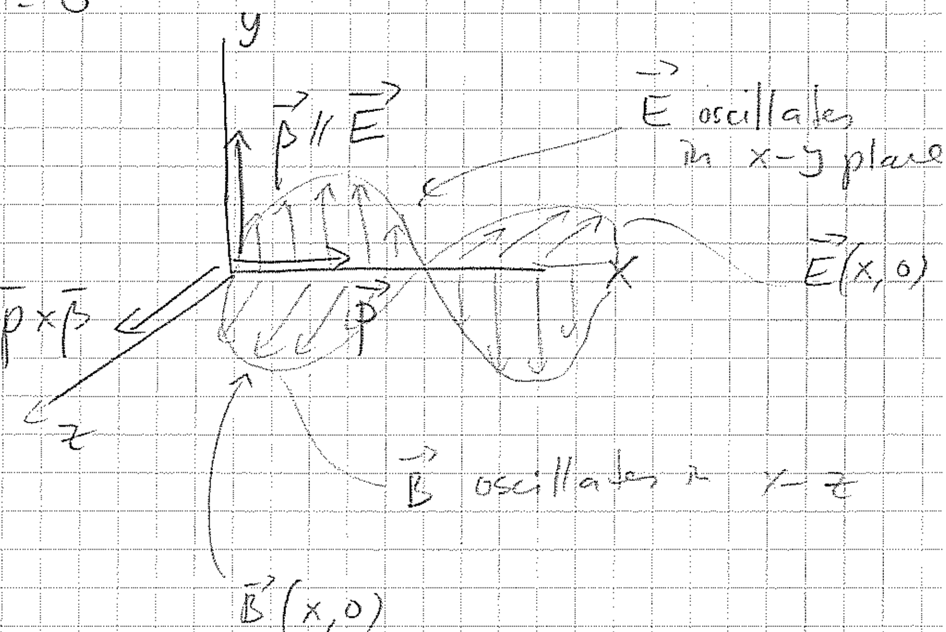
$$\vec{E} = p\vec{\beta}\,\sin(p(ct-x))$$

$$\vec{p}\cdot\vec{\beta} = 0$$

$(\vec{p}, \vec{E}, \vec{B})$ a r.h. basis

or





$ct - x = \text{const} =$

= surfaces of const. phase =

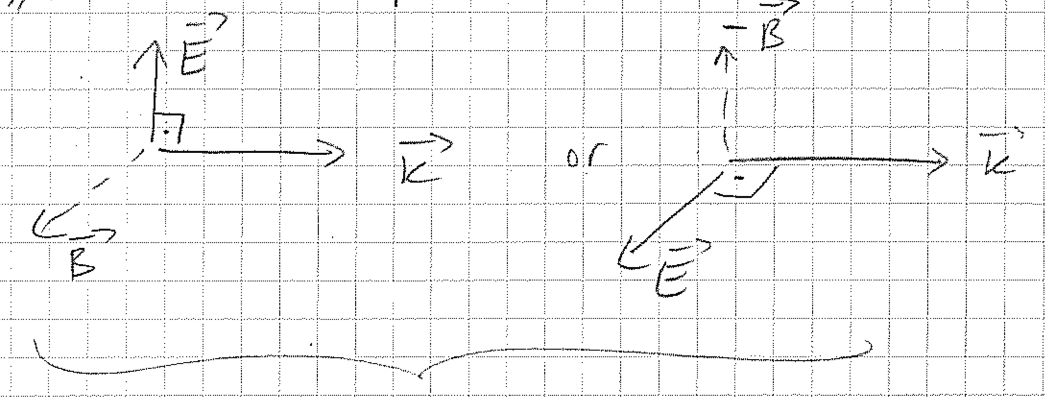
$\Rightarrow$ planes $\perp$ X axis $\rightarrow$

move to the right $(+x)$

w/ speed c.

Moral: the content of (121.1) is that general $\vec{A}(\vec{x},t)$ obeying free Maxwell equations is a superposition of plane waves labelled by wave vectors $\vec{k}$.

For every wave vector, there are two linearly independent choices of $\vec{\beta}$ $\Leftrightarrow$ two "polarizations"



\# these are called "linearly polarized" monochromatic EM waves

\# frequency $\omega_{\vec{k}} = c\,|\vec{k}|$; $T_{\vec{k}} = \frac{2\pi}{\omega_{\vec{k}}}$

\# wavelength $\lambda_{\vec{k}}$: $\sin(kct - kx)$

$x \to x + \frac{2\pi}{k} = x + \lambda$

phase repeats

$$\lambda_{\vec{k}} = \frac{2\pi}{|\vec{k}|} = \frac{2\pi c}{\omega_{\vec{k}}} = c\,T_{\vec{k}}.$$

▷ all the usual relations for waves hold.

As opposed to sound waves, EM waves are transverse : $\vec{E}$ & $\vec{B}$ oscillate $\perp$ $\vec{k}$.

(Transversality is a consequence of gauge invariance $(\vec{\nabla}\cdot\vec{A} = 0)$)

Just like sound waves <u>EM waves</u>:
<u>carry energy, momentum, angular momentum</u> —
— and <u>can transfer these to particles</u> w/ which field interacts — so we must learn about these now —

From (124.1) $\vec{A}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \left[e^{-i\omega_k t} \vec{\beta}(\vec{k}) + e^{i\omega_k t} \vec{\beta}^*(-\vec{k}) \right]$,

$$\vec{E}(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} |\vec{k}|\, e^{i\vec{k}\vec{x}} \left(i\vec{\beta}(\vec{k}) e^{-i\omega_k t} - i\vec{\beta}^*(-\vec{k}) e^{+i\omega_k t} \right),$$

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \left(i\vec{k}\times\vec{\beta}(\vec{k}) e^{-i\omega_k t} + i\vec{k}\times\vec{\beta}^*(-\vec{k}) e^{+i\omega_k t} \right),$$

(125.1) <u>We'll need these equations soon.</u>

But first, let's see what shall we call the energy & momentum of the EM field. The easiest road to this goal is to start from M. eqn:

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j} \quad \& \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$(\vec{E}\cdot)$ $\qquad$ $(\vec{B}\cdot)$

$$\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \frac{1}{c}\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j}\cdot\vec{E} \qquad \vec{B}\cdot(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}$$

subtract & rearrange

$$\frac{1}{c}\left(\vec{E}\frac{\partial\vec{E}}{\partial t} + \vec{B}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{4\pi}{c}\vec{j}\cdot\vec{E} + \underbrace{\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})}$$

$$+\vec{\nabla}\cdot(\vec{B}\times\vec{E}) = \vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{E})$$

$$\epsilon^{\alpha\beta\gamma}\partial_\alpha(B_\beta E_\gamma) = \epsilon^{\alpha\beta\gamma}(\partial_\alpha B_\beta)E_\gamma + \epsilon^{\alpha\beta\gamma}B_\beta \partial_\alpha E_\gamma$$

$$\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{d}{dt}\vec{E}^2 \quad \text{etc.} \quad \left(\text{no diff. } \frac{\partial}{\partial t} \text{ or } \frac{d}{dt}\right)$$

$$\frac{1}{2}\frac{1}{c}\frac{d}{dt}\left(\vec{E}^2 + \vec{B}^2\right) = -\frac{4\pi}{c}\vec{j}\cdot\vec{E} - \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)$$

$$\frac{d}{dt}\left(\frac{1}{8\pi}\left(\vec{E}^2+\vec{B}^2\right)\right) = -\vec{j}\cdot\vec{E} - \vec{\nabla}\cdot\underbrace{\left(\frac{c}{4\pi}\vec{E}\times\vec{B}\right)}$$

$$\frac{d}{dt}\left(\frac{1}{8\pi}\left(\vec{E}^2+\vec{B}^2\right)\right) = -\vec{j}\cdot\vec{E} - \vec{\nabla}\cdot\vec{S} \qquad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

"Poynting vector"

$S = \partial\mathcal{V}$

$\mathcal{V}$

fixed $\mathcal{V}$ in space

$$\frac{d}{dt}\left[\int_{\mathcal{V}} d^3x \left(\frac{\vec{E}^2+\vec{B}^2}{8\pi}\right)\right] = -\int_{\mathcal{V}} d^3x\, \vec{j}\cdot\vec{E} - \underbrace{\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{S}\, d^3x}_{}$$

$$= \oint d^2\vec{\sigma}\cdot\vec{S}$$

$S = \partial\mathcal{V}$

$d^2\vec{\sigma}$, $\vec{S}$

recall

$$\frac{d}{dt}\mathcal{E}_{kin} = e\,\vec{v}\cdot\vec{E}$$

(*)

& $\vec{j} = \vec{v}\rho$

$d^2\vec{\sigma}\cdot\vec{S}$ = flux of $\vec{S}$ thru $d^2\vec{\sigma}$

$$\frac{d}{dt}\left(\mathcal{E}_{\substack{EM\\ \text{field in } \mathcal{V}}}\right) = -\frac{d}{dt}\left(\mathcal{E}_{\substack{kin\\ \text{particles}\\ \text{in } \mathcal{V}}}\right) - \oint d^2\vec{\sigma}\cdot\vec{S}$$

$\partial\mathcal{V} = S$

will follow from $E = \frac{\partial L}{\partial v_i} v_i - L$ ... standard t... EM

(*) this is our interpretation of this term; that $\frac{\vec{E}^2+\vec{B}^2}{8\pi}$ = energy density of EM field

$$\frac{d}{dt}\left(\underset{\text{in } \mathcal{V}}{\mathcal{E}_{EM}} + \underset{\substack{\text{particles}\\ \text{in } \mathcal{V}}}{\mathcal{E}_{kin}} \right) = - \underbrace{\oint_{S=\partial\mathcal{V}} d^2\vec{\sigma}\cdot\vec{S}}$$

amount of energy leaving $\mathcal{V}$ in unit time.

hence $\underline{\underline{\vec{S}}}$ =

$= \underline{\text{(energy flux)}} = \left(\begin{array}{l}\text{amount of energy going through}\\ \text{an unit area } \perp \vec{S}\\ \text{in unit time}\end{array}\right)$

interpretation of Poynting vector.

<u>So, we learned:</u>

$$\frac{\vec{E}^2 + \vec{B}^2}{8\pi} = \text{energy density of EM field}$$

(very sensible from above – but not derived from symmetries (yet))

$$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{B} = \text{energy flux of EM field}$$

both taken at given $(\bar{x}, t)$.

Can be integrated over volume/surface to find integrated values...

?about EM waves?