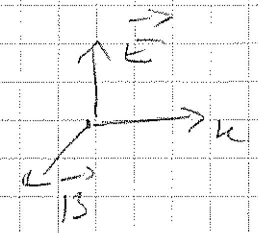
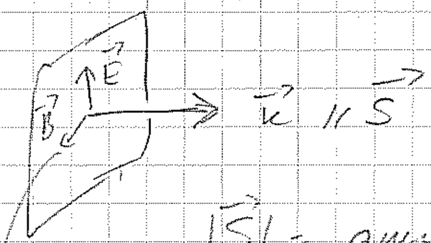
In a plane wave, since



clearly we have $\vec{k} \parallel \vec{S}$, so the energy flux in plane wave is in $\vec{k}$ direction



area A

$|\vec{S}|$ = amount of energy flowing through unit A in unit time

Also, in the wave $|\vec{E}| = |\vec{B}|$, $\vec{E} \perp \vec{B}$

so $\vec{S} = \frac{\vec{k}}{|\vec{k}|} \frac{c}{4\pi} \vec{E}^2$ but $\mathcal{E} \equiv \left(\frac{\text{energy}}{\text{unit volume}}\right) = \frac{\bar{E}^2 + \bar{B}^2}{8\pi} = \frac{\bar{E}^2}{4\pi}$

so $\vec{S} = \frac{\vec{k}}{|\vec{k}|} c \, \mathcal{E}$

energy per unit volume in wave

direction of wave propagation

speed of wave

energy flux — $[\vec{S}] = \frac{\text{energy}}{\text{time} \times \text{area}} = \frac{\text{momentum} \times \text{speed}}{\text{time} \times \text{area}}$

therefore → $\left[\frac{\vec{S}}{c^2}\right] = \frac{\text{momentum} \times \text{speed}}{\text{time} \times \text{area} \times (\text{speed})^2} = \frac{\text{momentum}}{\text{volume}}$

therefore it is natural to call

$\frac{\vec{S}}{c^2}$ = momentum density of the EM field
(momentum per unit volume!

In a treatment based on symmetries & Noether's theorem, § 32 & § 33 in L&L v. II,

(see for yourself)

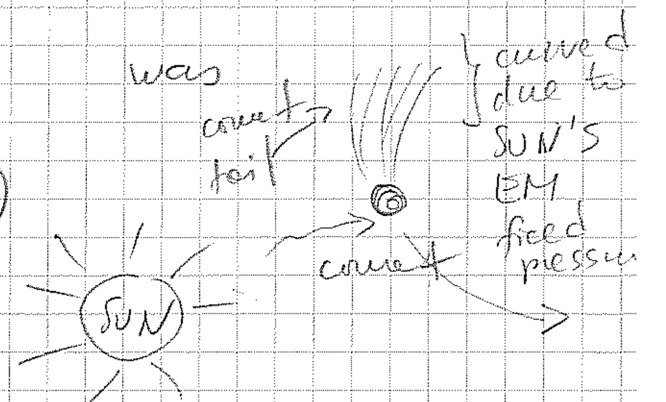
one finds that the symmetries of the action of the EM field under space & time translation lead to a host of conserved quantities — the energy momentum tensor of the EM field, T^{ij}

the energy density $\mathcal{E}$ & the momentum density, $\vec{S}/c^2$ are the T^{00} & $T^{0\alpha}$ components of that tensor —

[we may discuss this later in more detail!!]. (after radiation)

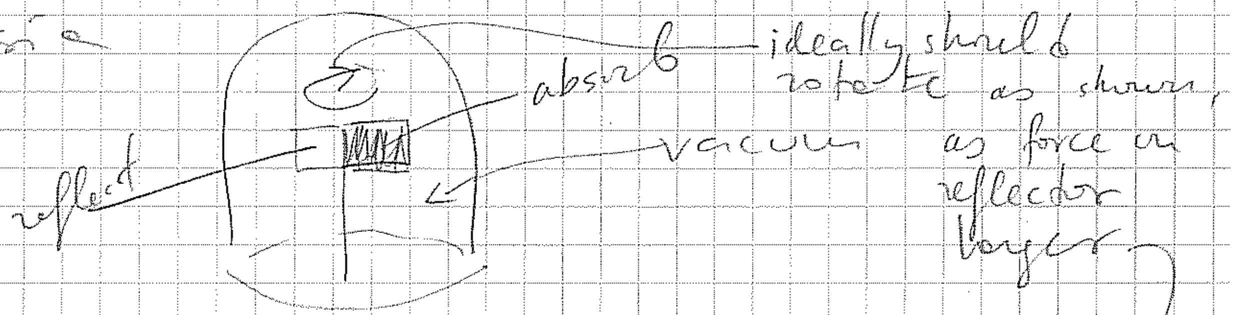
also note: $\mathcal{E} = c \left| \frac{\vec{S}}{c^2} \right|$ energy density $= c \times$ | momentum density | cf. for massless particles $E = c|\vec{p}|$

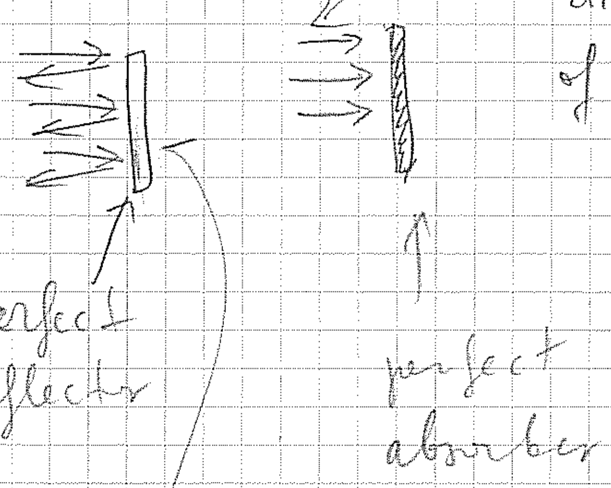
the fact that EM field carries momentum & can thus exert force was anticipated by Kepler (!)



& Maxwell

& was found, in 1900, by Peter Lebedev in Russia





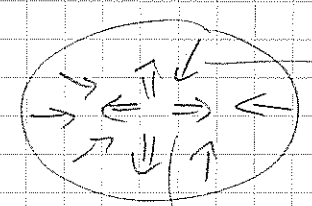
twice as much momentum (wrt perfect absorber) transferred to perfect reflector.

(need to cancel; radiation pressure of normal sources is tiny !! - see HW4)

but other effects to worry about ...

Just a few more words on the role of radiation pressure in physics, the universe, etc.

* already mentioned comet tails
* pressure of EM 'blackbody' radiation crucial for equilibrium of stars



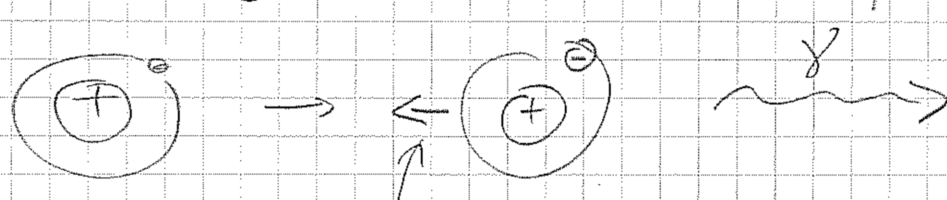
gravity – collapse

radiation pressure (& others) – expand

* in the Early Universe, expansion was driven by radiation (energy &) pressure in the "Radiation Dominated" epoch after Big Bang (from the "creation" to redshift ~3000 -- before CMB decoupling)

(or universe was 70k yrs old)

* the fact that EM waves' quanta – photons – carry momentum is ubiquitous --



recoil momentum

Our next topic is the fields created by moving charges & EM radiation ...

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the HUGE field of EM wave propagation thru media, reflection, & refraction, interference & diffraction — left for other courses & self study ...

We need equations that'll tell us how to determine fields created by charges ... 4-currents

clearly, we need to solve

$$\partial_k F^{ik} = -\frac{4\pi}{c} j^i$$

i.e. charge density & charge current of charged particles — assumed given here

As usual the other

$$\epsilon^{klmn} \partial_l F_{mn} = 0$$

is solved by

$$F_{mn} = \partial_m A_n - \partial_n A_m \; ; \text{ so}$$

use this:

$$\partial_k F^{ik} = -\frac{4\pi}{c} j^i \implies$$

$$\partial_k(\partial^i A^k - \partial^k A^i) = -\frac{4\pi}{c} j^i .$$

$$\partial^i(\partial_k A^k) - \partial_k \partial^k A^i = -\frac{4\pi}{c} j^i \quad (*)$$

here, it is especially convenient to use Lorentz gauge, i.e. $\partial_k A^k = 0$, because then equ. is simple.

$$\partial_k \partial^k A^i = \frac{4\pi}{c} j^i$$

denote $\partial_k \partial^k = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \equiv \Box$

Laplac-ian

D'Alembert-ian or D'Alembert's operator or wave operator

& we have

$$\Box A^i = \frac{4\pi}{c} j^i$$

Qu-n: what if we had used Coulomb gauge ?

$(*)$ holds always $\rightarrow$ Coulomb gauge $A^0 = 0$, so we'd have

$$\left.\begin{array}{l} i=0 \\ \text{of } (*) \end{array}\right\} \Rightarrow \partial^0(-\vec{\partial}\cdot\vec{A}) = -4\pi\rho \Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho \quad (*.1)$$

$$\left.\begin{array}{l} i=\alpha \\ \text{of } (*) \end{array}\right\} \Rightarrow \vec{\nabla}(-\vec{\partial}\cdot\vec{A}) - \Box\vec{A} = -\frac{4\pi}{c}\vec{j} \quad (*.2)$$

(*.1), when $\rho = 0$, was used to argue that

i.e. in vacuum

$\partial_0 (\bar{\nabla} \cdot \bar{A}) = 0$ & enabled us to further choose a gauge, via a time-independent $\chi(\vec{x})$ [s.t. $A^0 = 0$ preserved) s.t. $\bar{\nabla} \cdot \bar{A} = 0$ ⟶ <u>here we can't do this</u>!

→ which makes Coulomb gauge not very convenient to study radiation →

A bit of moral: while all "gauges" are equivalent, depending on problem at hand, there are "better" or "worse" ones ...

although it was convenient to study waves – and establish transversality etc.. – & of course, since those results only involve E & B, they hold irrespective of the fact that they were derived in Coulomb gauge (could've done Lorentz ...)

Back to our equ. from p. 133 :

(Lorentz gauge recall:

$$\Box A^i = \frac{4\pi}{c} j^i \qquad \left(\left(\partial_i A^i = 0;\ A^i \to A^i + \partial^i \chi \text{ w/ } \Box \chi = 0 \text{ still allowed gauge transformation} \right)\right)$$

wave equ. w/ a "source" j^i

There are really 4 eqns, but their forms are identical which makes things easy...

So we need to learn to solve eqns like this

$$\Box f(x) = g(x)$$

Reminder:

You've seen things like $\Delta f(\vec{r}) = g(\vec{r})$

~ potential due to density

~ charge density

& you know that using superposition principle, it suffices to find the field of a point charge $g(\vec{r}) = g_0 \, \delta^{(3)}(\vec{r} - \vec{r}_0)$, Call it

$G(\vec{r}, \vec{r}_0)$ i.e. $\Delta_{\vec{r}} G(\vec{r}, \vec{r}_0) = \delta^{(3)}(\vec{r} - \vec{r}_0) \rightarrow$

$\rightarrow$ then if we know $G(\vec{r}, \vec{r}_0)$, we can construct

$f(\vec{r}) = \int d^3 r_0 \, G(\vec{r}, \vec{r}_0) \, g(\vec{r}_0)$, which solves

$\Delta f(\vec{r}) = g(\vec{r})$ since $\Delta f(\vec{r}) = \int d^3 r_0 \, \Delta_{\vec{r}} G(\vec{r}, \vec{r}_0) \, g(\vec{r}_0)$

$= \int d^3 r_0 \, \delta^{(3)}(\vec{r} - \vec{r}_0) \, g(\vec{r}_0) = g(\vec{r})$ ■.

Goal: • Repeat for $\Box f(x) = g(x)$.

• pay attention to boundary conditions & causality.