Before we continue, reminder about Laplace equ's Green's function:

$$\Delta_{\vec{x}}\, G(\vec{x}-\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$$

has a solution $G(\vec{x}-\vec{x}') = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}$

In other words, the potential due to a charge density $\rho(\vec{x})$, which obeys

$$\Box A^0 = \frac{4\pi}{c} j^0 = 4\pi\rho(\vec{x})$$

$$\parallel$$

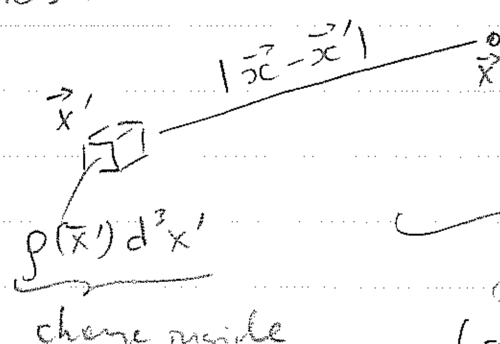
$$-\Delta\varphi(\vec{x}) = 4\pi\rho(\vec{x})$$

or $\Delta\varphi(\vec{x}) = -4\pi\rho(\vec{x})$ can be found

as $$\varphi(\vec{x}) = \int d^3\vec{x}'\, \frac{\rho(\vec{x}')}{|\vec{x}'-\vec{x}'|} \qquad (*)$$

NB: (1) No 4π, no ϵ_0 in our units ["Gaussian"]

(2) physics:



$$\varphi(\vec{x}) = \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

then $\int$ over all $\vec{x}'$! (superposition principle).

(3) (*) is the basis of the <u>multipole expansion</u>

region where $\rho \neq 0$

• $\varphi(\text{outside}) = \frac{\text{charge}}{r} + \frac{\text{dipole}}{r^2} + \frac{\text{quadrup}}{r^3} \ldots$

We'll proceed similarly now.
We have to solve

$$\Box A^i = \frac{4\pi}{c} j^i$$

Consider $i=0$, since in Lorentz gauge all eqns. same, to get the rest simply replace $0 \to i$.

Of course $j^0 = j^0(\vec{x}, t)$, (remember $j^0 = c\rho$)

Imagine now that we can solve

$$\Box G(\vec{x}, t) = \delta^{(3)}(\vec{x})\, \delta(t)$$

(roughly, a charge density which is $\neq$ only at $\vec{x}=0$ and at $t=0$ (charge exists for an instance at a point).

then also

$$\Box_{(\bar{x},t)} G(\bar{x}-\bar{x}', t-t') = \delta^{(3)}(\bar{x}-\bar{x}')\, \delta(t-t')$$

since, it doesn't matter where charge is

$$\left(\Box_{\bar{x},t} \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta_{\bar{x}} \right)$$

If $G(\bar{x}-\bar{x}', t-t')$ is known, then we can solve for A^0:

(over all spacetime)

$$A^0(\bar{x}, t) = \int d^3x'\, d^3t'\, \frac{4\pi}{c} j^0(\bar{x}', t')\, G(\bar{x}-\bar{x}', t-t')$$

since $\Box_{\bar{x},t} A^0 = \int d^3x'\, dt'\, \delta^3(\bar{x}-\bar{x}')\, \delta(t-t')\, \frac{4\pi}{c} j^0(\bar{x}', t') = \frac{4\pi}{c} j^0(\bar{x}, t)$ ∎

So what is $G(\bar{x}, t)$?

<u>Claim:</u> $$G(\bar{x}, t) \equiv \frac{\delta(t - |\vec{x}|/c)}{4\pi |\vec{x}|} \qquad (**)$$

is the so called "retarded" Green's function of the D'Alembert equation, i.e. it obeys
$$\Box G = \delta^{(3)}(\bar{x})\, \delta(t)$$ & gives the

(minus) <u>retarded potential</u> due to a unit point charge at $\bar{x} = 0$ existing at $t = 0$ only.

Let's explain: (N.B. there's a systematic way of obtaining G for D'Alembert's equation - using Fourier transforms and residue calculus (complex variables). We'll be happy by showing that $(**)$ is the solution)

First, let's show that $G(\bar{x}, t)$ $(**)$ obeys
$$\Box G = \delta^{(3)}(\bar{x})\, \delta(t)$$. Simple - no brainer -

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \frac{\delta(t - |\vec{x}|/c)}{4\pi|\vec{x}|} = \frac{1}{c^2 4\pi|\vec{x}|} \delta''\left(t - \frac{|\vec{x}|}{c}\right) \qquad (**.1)$$

$$\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{4\pi|\vec{x}|}\delta\left(t - \frac{|\vec{x}|}{c}\right)\right) = \underbrace{\left(\vec{\nabla}^2 \frac{1}{4\pi|\vec{x}|}\right)}_{= -\delta^{(3)}(\vec{x})} \delta\left(t - \frac{|\vec{x}|}{c}\right) + \longrightarrow$$

(remember electrostatics, or p. 136)

$$+\frac{1}{4\pi|\vec{x}|}\vec{\nabla}^2\delta\left(t-\frac{|\vec{x}|}{c}\right)+\frac{2}{4\pi}\vec{\nabla}\frac{1}{4\pi|\vec{x}|}\cdot\vec{\nabla}\delta\left(t-\frac{|\vec{x}|}{c}\right)$$ 159

$$=-\delta^{(3)}(\vec{x})\,\delta\left(t-\frac{|\vec{x}|}{c}\right)+\underbrace{\frac{1}{4\pi|\vec{x}|}\vec{\nabla}^2\delta\left(t-\frac{|\vec{x}|}{c}\right)+\frac{2}{4\pi}\vec{\nabla}\frac{1}{|\vec{x}|}\cdot\vec{\nabla}\delta\left(t-\frac{\vec{x}}{c}\right)}=[$$

$$\frac{1}{4\pi|\vec{x}|}\vec{\nabla}\left[\delta'\left(t-\frac{|\vec{x}|}{c}\right)\left(-\frac{1}{c}\right)\vec{\nabla}|\vec{x}|\right]+\frac{2}{4\pi}\left(-\frac{1}{|\vec{x}|^2}\right)\vec{\nabla}|\vec{x}|\cdot\vec{\nabla}|\vec{x}|\left(-\frac{1}{c}\right)\times\delta'\left(t-\frac{|\vec{x}|}{c}\right)$$

$$=-\frac{1}{4\pi c|\vec{x}|}\delta''\left(t-\frac{|\vec{x}|}{c}\right)\left(-\frac{1}{c}\right)\vec{\nabla}|\vec{x}|\cdot\vec{\nabla}|\vec{x}|-\frac{1}{4\pi c|\vec{x}|}\delta'\left(t-\frac{|\vec{x}|}{c}\right)\Delta|\vec{x}|$$

$$+\frac{1}{2\pi c|\vec{x}|^2}\delta'\left(t-\frac{|\vec{x}|}{c}\right)\vec{\nabla}|\vec{x}|\cdot\vec{\nabla}|\vec{x}|=$$

use: $\Delta|\vec{x}|=\frac{2}{|\vec{x}|}$

since $\vec{\nabla}|\vec{x}|=\frac{\vec{x}}{|\vec{x}|}$

$$\vec{\nabla}^2|\vec{x}|=\vec{\nabla}\frac{\vec{x}}{|\vec{x}|}=\frac{\vec{\nabla}\cdot\vec{x}}{|\vec{x}|}+\vec{x}\cdot\vec{\nabla}\frac{1}{|\vec{x}|}$$

$$=\frac{3}{|\vec{x}|}-\frac{\vec{x}\cdot\vec{\nabla}|\vec{x}|}{|\vec{x}|^2}=\frac{2}{|\vec{x}|}$$

& $\left(\vec{\nabla}|\vec{x}|\right)^2=1$

$$=+\frac{1}{4\pi c^2}\frac{1}{|\vec{x}|}\delta''\left(t-\frac{|\vec{x}|}{c}\right)$$

$$\cancel{-\frac{1}{2\pi c|\vec{x}|^2}\delta'\left(t-\frac{|\vec{x}|}{c}\right)}$$

$$\cancel{+\frac{1}{2\pi c|\vec{x}|^2}\delta'\left(t-\frac{|\vec{x}|}{c}\right)}$$

$$=\frac{1}{4\pi c^2|\vec{x}|}\delta''\left(t-\frac{|\vec{x}|}{c}\right)$$

so $\boxed{***}=\Delta_{\vec{x}}\left(\frac{1}{4\pi|\vec{x}|}\delta\left(t-\frac{|\vec{x}|}{c}\right)\right)=-\delta^{(3)}(\vec{x})\,\delta\left(t-\frac{|\vec{x}|}{c}\right)+\frac{\delta''\left(t-\frac{|\vec{x}|}{c}\right)}{4\pi c^2|\vec{x}|}$

& $(**\cdot 1)$ $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{4\pi|\vec{x}|}\delta\left(t-\frac{|\vec{x}|}{c}\right)=\frac{1}{4\pi c^2|\vec{x}|}\delta''\left(t-\frac{|\vec{x}|}{c}\right)$

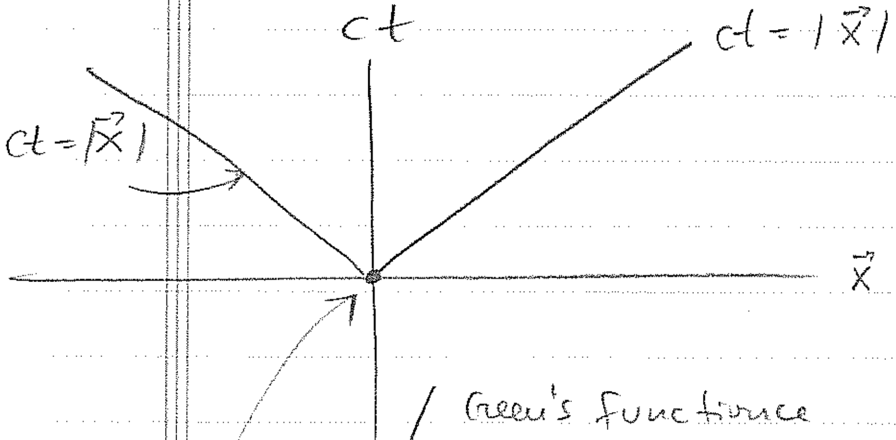
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta_{\vec{x}}\right)\left(\frac{1}{4\pi|\vec{x}|}\delta\left(t-\frac{|\vec{x}|}{c}\right)\right)=+\delta^{(3)}(\vec{x})\,\delta\left(t-\frac{\vec{x}}{c}\right)=\delta^{(3)}(\vec{x})\,\delta(t)$$ ∎

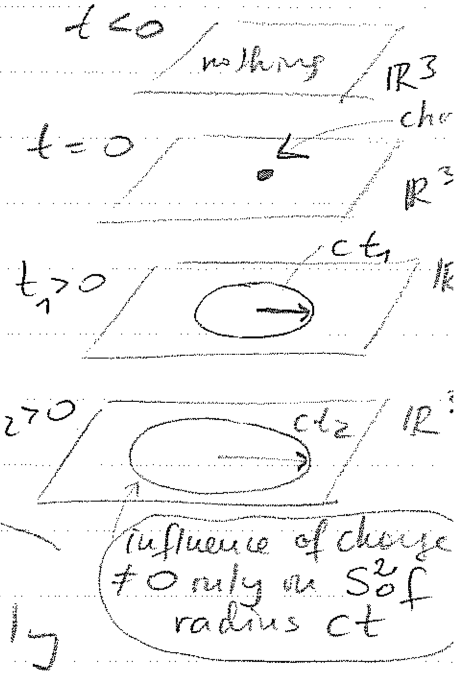
Hence, we've shown that indeed

$$\Box \, \frac{\delta\left(t - \frac{|\bar{x}|}{c}\right)}{4\pi |\bar{x}|} = \delta^{(3)}(\bar{x})\, \delta(t)$$

as required for $G(\bar{x}, t)$.

Why is this Green's function "retarded"?





(Green's functions used to be called "influence functions")

our "charge" existed at $t=0$, $\bar{x}=0$ only

its "potential", $G(t, \vec{x})$, is $\sim \delta\left(t - \frac{|\bar{x}|}{c}\right)$

so it is $\neq 0$ only on the light-cone in the future

but it vanishes at $ct = -|\vec{x}|/c$, i.e. on the past light cone.

In other words the influence of the charge @ $t=0$, $\bar{x}=0$ is only felt in the future. It is only felt on the light cone, because light (EM field of charge) travels w/ speed c.

A Green's function $G(\bar{x},t) = \dfrac{\delta(|\bar{x}|/c + t)}{4\pi|\bar{x}|}$

would be $\neq 0$ only on past light cone. This function is called "advanced" - because the influence of the charge would be only felt in the past.

In classical EM we only deal w/ retarded Green's function - where the effect of the cause (EM field of charge) is only felt after the charge's "creation".

Back to our $\Box A^i(\bar{x},t) = \frac{4\pi}{c} j^i(\bar{x},t)$ - we can solve by

$$A^i(\bar{x},t) = \frac{1}{c}\underbrace{\int}_{\text{(all spacetime)}} d^3\vec{x}'\,dt'\ \frac{\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) j^i(\vec{x}',t')}{|\vec{x}-\vec{x}'|} + \tag{171.1}$$

$$+ \begin{pmatrix}\text{a solution of the homogeneous equation} \\ \equiv \sum \text{of EM waves, arbitrary}\end{pmatrix}$$

<u>Proof</u>: $\Box_{\bar{x},t} A^i(\bar{x},t) = \frac{4\pi}{c}\int d^3x'dt'\ \underbrace{\Box_{\bar{x},t}\left(\frac{\delta\left(t-t'-\frac{|\bar{x}-\bar{x}'|}{c}\right)}{4\pi|\bar{x}-\bar{x}'|}\right)}_{\delta^{(3)}(\bar{x}-\bar{x}')\,\delta(t-t')} j^i(\bar{x}',t')$

$= \frac{4\pi}{c} j^i(\bar{x},t)$ ∎

In the general retarded solution of nonhomogeneous equations, we can easily take t' integral - this is because we can express $t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$,

hence solution really is

$$A^i(\bar{x},t) = \frac{1}{c}\int d^3x' \, \frac{j^i(\vec{x}',\, t - \frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|} \qquad (142.1)$$

i.e 4-potentials @ $(\bar{x},t)$ only depend on the 4-current densities at $\underbrace{t - \frac{|\bar{x}-\bar{x}'|}{c}}$

This is all very clear physically (this is called the "retarded time") especially if we consider the EM potentials due to a charged particle moving on some worldline:

let us parametrize the worldline as: $\vec{x}_c(t)$

the charge density is $\rho(\vec{x},t) = e\, \delta^{(3)}(\vec{x} - \vec{x}_c(t))$

charge of particle

density is $\neq 0$ only on worldline & $\int d^3\bar{x}$ gives e $\forall t$, as it should.

and current is, since $\vec{v}_c(t) = \dot{\vec{x}}_c(t)$,

$$\left\{\begin{aligned} \vec{j}(\vec{x},t) &= e\, \dot{\vec{x}}_c(t)\, \delta^{(3)}(\vec{x}-\vec{x}_c(t)) \\ j^0(\vec{x},t) &= c\, e\, \delta^{(3)}(\vec{x}-\vec{x}_c(t)) \end{aligned}\right\} \rightarrow \text{4-current of charged particle}$$

(recall p. 100 & before)

then, use (141.1) to find $A^i(\bar{x},t)$:

$$\begin{pmatrix} A^0(\bar{x},t) \\ \vec{A}(\vec{x},t) \end{pmatrix} = \frac{1}{c}\int d^3x'\,dt' \, \frac{\delta\left(t - t' - \frac{|\bar{x}-\bar{x}'|}{c}\right)}{|\bar{x}-\bar{x}'|} \begin{pmatrix} ce \\ e\,\dot{\vec{x}}_c(t') \end{pmatrix} \times$$

$$\times \, \delta^{(3)}(\vec{x}' - \vec{x}_c(t'))$$

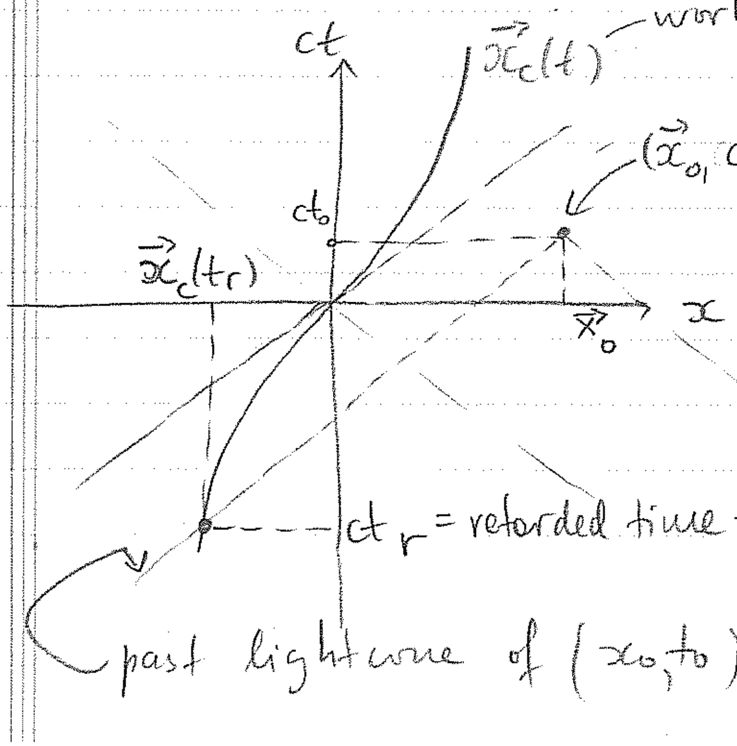
$$\left[\begin{pmatrix} j^0 \\ \vec{j} \end{pmatrix} (\vec{x}', t') \right]$$

use $\delta^{(3)}(\vec{x}' - \vec{x}_c(t'))$
to replace $\vec{x}' \to \vec{x}_c(t')$
and drop d^3x':

$$\begin{pmatrix} A^0(\bar{x},t) \\ \vec{A}(\bar{x},t) \end{pmatrix} = e\int dt' \begin{pmatrix} 1 \\ \frac{\dot{\vec{x}}_c(t')}{c} \end{pmatrix} \frac{\delta\left(t - t' - \frac{|\vec{x}-\vec{x}_c(t')|}{c}\right)}{|\vec{x} - \vec{x}_c(t')|} \qquad (143.1)$$

what's this?

pick an arbitrary space time point



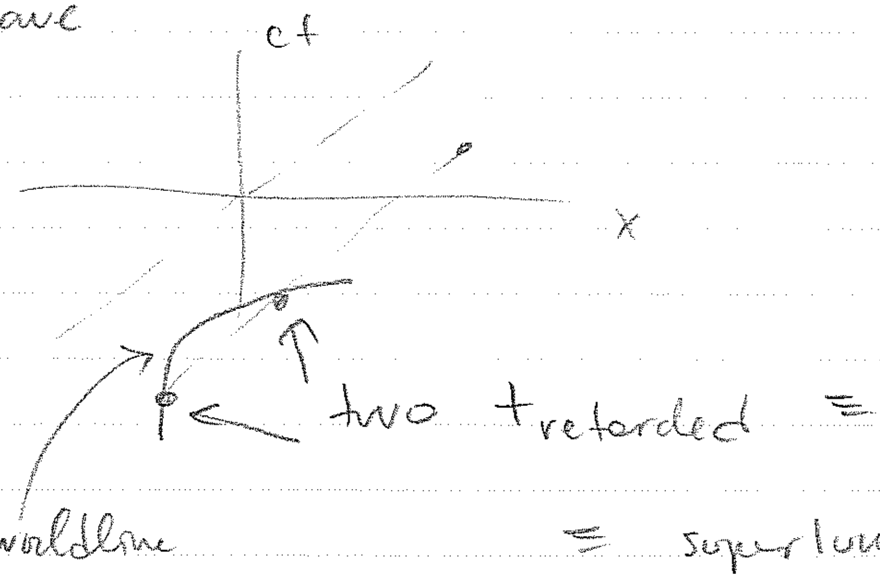
$(\vec{x}_0, ct_0)$: $A^i(x_0 t_0) = ?$: can only be influenced by EM fields created on the past light cone of (x^0, ct_0)

= intersection of past light cone of $(\bar{x}_0, ct_0)$ w/ worldline

past lightcone of (x_0, t_0)

Note: $(ct_r, \vec{x}_c(t_r))$ uniquely determined for any given $\vec{x}_0, ct_0$ —

– only one intersection – if not, we'd have



≡ superluminal motion in between.

From picture, it is clear that

$$c(t_r - t_0) = |\vec{x}_0 - \vec{x}_c(t_r)|$$

- this is the mathematical expression which one needs to solve to find t_{retarded}, given trajectory & $(\vec{x}_0, ct_0)$.
- as discussed, have unique solution

Hence: $\int dt'$ in (143.1) only receives a contribution for t' where argument of δ-fn is zero ⟶

which is exactly ("surprise" – I promised you that already on p. 142, see (142.1)) the retarded time.

Still not done, however — recall

$$\int dx \; \delta(f(x)) = \frac{\delta(x_0)}{|f'(x_0)|}$$

where $f(x_0) = 0$.

For us, $\int dx \to \int dt'$ in (143.1)

$$x_0 \equiv t_r \; ; \quad f'(x_0) = \frac{\partial}{\partial t'}\left(t - t' - \frac{1}{c}|\vec{x} - \vec{x}_c(t')|\right) \text{ at } t' = t_r .$$

Let's do it.

$$\begin{pmatrix} A^0 \\ \vec{A} \end{pmatrix}(\vec{x}, t) = e\int_{-\infty}^{\infty} dt' \begin{pmatrix} 1 \\ \frac{\dot{\vec{x}}_c(t')}{c} \end{pmatrix} \frac{\delta\left(t - t' - \frac{1}{c}|\vec{x} - \vec{x}_c(t')|\right)}{|\vec{x} - \vec{x}_c(t')|} = (***)$$

$$f'(t_r) = \frac{\partial}{\partial t'}\left(t - t' - \frac{|\vec{x} - \vec{x}_c(t')|}{c}\right)\Bigg|_{t' = t_r} =$$

$$= -1 - \frac{1}{c}\frac{\partial}{\partial t'}\sqrt{(\vec{x} - \vec{x}_c(t'))\cdot(\vec{x} - \vec{x}_c(t'))}\,\Bigg|_{t' = t_r}$$

$$= -1 - \frac{1}{c}\frac{1}{2\sqrt{(\vec{x} - \vec{x}_c(t'))^2}}\, 2(\vec{x} - \vec{x}_c(t'))\cdot\Big(-\underbrace{\frac{\partial}{\partial t'}\vec{x}_c(t')}_{\vec{v}_c(t_r)}\Big)\Bigg|_{t' = t_r}$$

$$= -1 + \frac{1}{c}\frac{(\vec{x} - \vec{x}_c(t_r))\cdot\vec{v}_c(t_r)}{|\vec{x} - \vec{x}_c(t_r)|} = \longrightarrow$$

$$= \frac{\vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r)) - c\,|\vec{x}-\vec{x}_c(t_r)|}{c\,|\vec{x}-\vec{x}_c(t_r)|}$$

remember this was $f'(t_r)$

Hence $\boxed{**\,*} = \begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix}(\vec{x},t) = e \begin{pmatrix} 1 \\ \frac{\vec{v}_c(t_r)}{c} \end{pmatrix} \frac{c\,\cancel{|\vec{x}-\vec{x}_c(t_r)|}}{|\vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r)) - c\,|\vec{x}-\vec{x}_c(t_r)||}$

$$\times \cancel{\frac{1}{|\vec{x}-\vec{x}_c(t_r)|}} =$$

$$\begin{pmatrix} A_0 \\ \vec{A} \end{pmatrix}(\vec{x},t) = \begin{pmatrix} \dfrac{e\,c}{|\vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r)) - c\,|\vec{x}-\vec{x}_c(t_r)||} \\[2ex] \dfrac{e\,\vec{v}_c(t_r)}{|\vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r)) - c\,|\vec{x}-\vec{x}_c(t_r)||} \end{pmatrix}$$

finally $\vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r)) \leqslant c\,|\vec{x}-\vec{x}_c(t_r)|$

hence

$$\begin{cases} A^0(\vec{x},t) = \dfrac{e\,c}{c\,|\vec{x}-\vec{x}_c(t_r)| - \vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r))} \\[3ex] \vec{A}(\vec{x},t) = \dfrac{e\,\vec{v}_c(t_r)}{c\,|\vec{x}-\vec{x}_c(t_r)| - \vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r))} \end{cases}$$

"Lienard-Wiechert potentials"

$\Longrightarrow$ simple check: $\vec{v}_c = 0$; $\vec{x}_c =$ const: $A^0 = \dfrac{e}{|\vec{x}-\vec{x}_c|}$, $\vec{A} = 0$

((— as should be))