We need $\vec{E}$, $\vec{B}$, but only have

$$A^0(\vec{x},t) = \frac{ec}{c\,|\vec{x}-\vec{x}_c(t_r)| - \vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r))}$$

&

$$\vec{A}(\vec{x},t) = \frac{\vec{v}_c(t_r)}{c}\,A^0(\vec{x},t)$$

w/ $\quad c(t-t_r) = |\vec{x}-\vec{x}_c(t_r)|$

to find $\vec{E}$ & $\vec{B}$ $\Rightarrow$ $\vec{E} = -\frac{\partial}{\partial\vec{x}}A^0(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}(\vec{x},t)$

&

$$\vec{B} = \frac{\partial}{\partial\vec{x}} \times \vec{A}(\vec{x},t)$$

all is well, but $t_r = t_r(\vec{x},t)$: in

HW 5 (you) will show that

$$\frac{\partial t_r(\vec{x},t)}{\partial t} = \frac{c\,|\vec{x}-\vec{x}_c(t_r)|}{c\,|\vec{x}-\vec{x}_c(t_r)| - \vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r))}$$

&

$$\frac{\partial}{\partial\vec{x}}\Big(t_r(\vec{x},t)\Big) = -\frac{\vec{x}-\vec{x}_c(t_r)}{c\,|\vec{x}-\vec{x}_c(t_r)| - \vec{v}_c(t_r)\cdot(\vec{x}-\vec{x}_c(t_r))}$$

& then do all differentiations to get $\vec{E}$ & $\vec{B}$

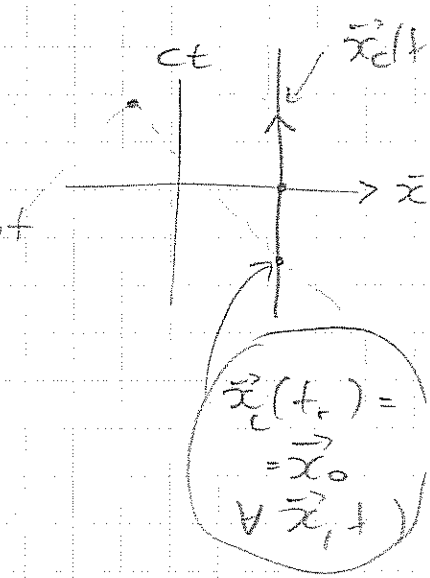
after some work, the result is

$$\vec{E}(\vec{x},t) = e\,\frac{|\vec{x}-\vec{x}_c(t_r)|}{\left((\vec{x}-\vec{x}_c(t_r))\cdot\vec{u}\right)^3}\Big[\left(c^2-|\dot{\vec{x}}_c(t_r)|\right)\vec{u} + \left(\vec{x}-\vec{x}_c(t_r)\right)\times\left(\vec{u}\times\ddot{\vec{x}}_c(t_r)\right)\Big]$$

$$\vec{B}(\vec{x},t) = \frac{\vec{x}-\vec{x}_c(t_r)}{|\vec{x}-\vec{x}_c(t_r)|}\times\vec{E}(\vec{x},t)$$

where $\quad \vec{u} = c\,\dfrac{\vec{x}-\vec{x}_c(t_r)}{|\vec{x}-\vec{x}_c(t_r)|} - \dot{\vec{x}}_c(t_r)$

-- looks messy -- checks:



(1) let $\vec{x}_c(t) = \vec{x}_0 = $ const — particle @ rest

$\vec{x}_c(t_r) = \vec{x}_0$ as well, $\quad \dot{\vec{x}}_c = \ddot{\vec{x}}_c = 0$

$$\vec{u} = c\,\frac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|}$$

$$(\vec{x}-\vec{x}_0)\cdot\vec{u} = c\,|\vec{x}-\vec{x}_0|$$

so $$\vec{E}(\vec{x},t) = e\,\frac{|\vec{x}-\vec{x}_0|}{c^3|\vec{x}-\vec{x}_0|^3}\,c^2 c\,\frac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|} = e\,\frac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|^3}$$

$$\vec{B}(\vec{x},t) = 0 \quad \left((\vec{x}-\vec{x}_0)\times(\vec{x}-\vec{x}_0) = 0\right).$$

(2) - more interesting →

→ particle in uniform motion, $\vec{x}_c = \vec{v}\,t$ --- easier to do starting from A^0, $\vec{A}$ (from $\vec{E}$ & $\vec{B}$ above, HW5) we have that $|\vec{x} - \vec{v}\,t_r| = c(t - t_r)$

hence $(\vec{x} - \vec{v}t_r)^2 = c^2(t - t_r)^2$

$$\vec{x}^2 - 2\vec{x}\cdot\vec{v}\,t_r + t_r^2 v^2 = c^2t^2 - 2c^2t\,t_r + t_r^2c^2$$

$$t_r^2(c^2 - v^2) - 2(c^2t - \vec{x}\cdot\vec{v})\,t_r + (c^2t^2 - \vec{x}^2) = 0$$

$$t_r = \frac{c^2t - \vec{x}\cdot\vec{v} \pm \sqrt{(c^2t - \vec{x}\,\vec{v})^2 - (c^2 - v^2)(c^2t^2 - \vec{x}^2)}}{c^2 - v^2}$$

ambiguous sign? at $\vec{v} = 0$ must have $t_r = t - \frac{|\vec{x}|}{c}$

so $t_r\big|_{\vec{v}=0} = \dfrac{c^2t \pm \sqrt{\cancel{c^4t^2} - \cancel{c^4t^2} + c^2\vec{x}^2}}{c^2}$

$= t \pm \frac{|\vec{x}|}{c} \Rightarrow$ so sign is "$-$"

recall

$$A^0(\vec{x}, t) = \frac{ec}{c|\vec{x} - \vec{v}t_r| - \vec{v}\cdot(\vec{x} - \vec{v}t_r)} \quad \left(\; \& \; \vec{A} = \frac{\vec{v}}{c}A^0 \; \right)$$

$$= \frac{ec}{c^2(t - t_r) - \vec{v}\cdot\vec{x} + v^2t_r} = \frac{ec}{c^2t - c^2t_r - \vec{v}\cdot\vec{x} + \vec{v}^2t_r}$$

$$= \frac{ec}{c^2t - \vec{v}\cdot\vec{x} - (c^2 - v^2)t_r} = \frac{ec}{\cancel{c^2t} - \cancel{\vec{v}\cdot\vec{x}} - \cancel{tc^2} + \cancel{\vec{x}\vec{v}} + \sqrt{(c^2t - \vec{x}\cdot\vec{v})^2 - (c^2 - v^2)(c^2t^2 - \vec{x}^2)}}$$

plug in t_r

$$A^0(\vec{x}, t) = \frac{ec}{\sqrt{(c^2t - \vec{x}\cdot\vec{v})^2 - (c^2 - v^2)(c^2t^2 - \vec{x}^2)}} =$$

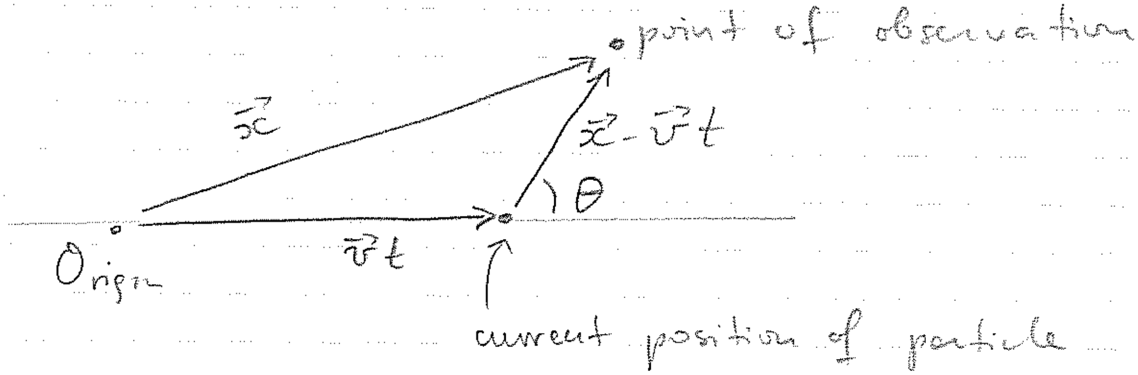
$$= \frac{ec}{(\cancel{c^4t^2} - 2c^2t(\vec{x}\cdot\vec{v}) + (\vec{x}\cdot\vec{v})^2 - \cancel{c^4t^2} + c^2\vec{x}^2 + v^2c^2t^2 - v^2\vec{x}^2)^{1/2}} =$$

$$= e\Big(\underbrace{\vec{x}^2 - 2t\,\vec{x}\cdot\vec{v} + \vec{v}^2t^2}_{(\vec{x} - \vec{v}t)^2} + \frac{(\vec{x}\cdot\vec{v})^2 - v^2\vec{x}^2}{c^2} \Big)^{-1/2} = \longrightarrow$$

$\underbrace{(\vec{x} - \vec{v}t)^2}$ vector between point of observation & current position of parti[cle]

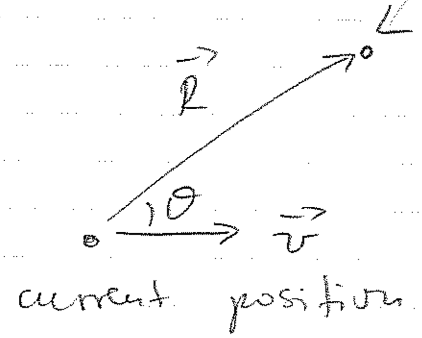
i.e., to moment of observation

so $$A^0(\bar{x},t) = \frac{e}{|\bar{x}-\bar{v}t|} \times \frac{1}{\sqrt{1-\frac{1}{c^2}\frac{v^2\vec{x}^2-(\vec{x}\cdot\vec{v})^2}{(\bar{x}-\bar{v}t)^2}}}$$



Claim: $$\frac{v^2x^2-(\bar{x}\cdot\vec{v})^2}{(\bar{x}-\bar{v}t)^2} = v^2\sin^2\theta$$

--- usually written as $$A^0(\bar{x},t) = \frac{e}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$$



(both $\vec{R}$ & θ depend on t, of course)

$$\vec{A} = \frac{\vec{v}}{c}A^0(\bar{x},t)$$

$$\cos\theta = \frac{(\vec{x}-\vec{v}t)\cdot\vec{v}}{|\bar{x}-\bar{v}t|\,v} = \frac{\vec{x}\cdot\vec{v} - v^2t}{|\bar{x}-\bar{v}t|\,v}$$

$$\sin^2\theta = 1-\cos^2\theta = \frac{v^2(\bar{x}-\bar{v}t)^2-(\bar{x}\cdot\bar{v}-v^2t)^2}{|\bar{x}-\bar{v}t|^2v^2} =$$

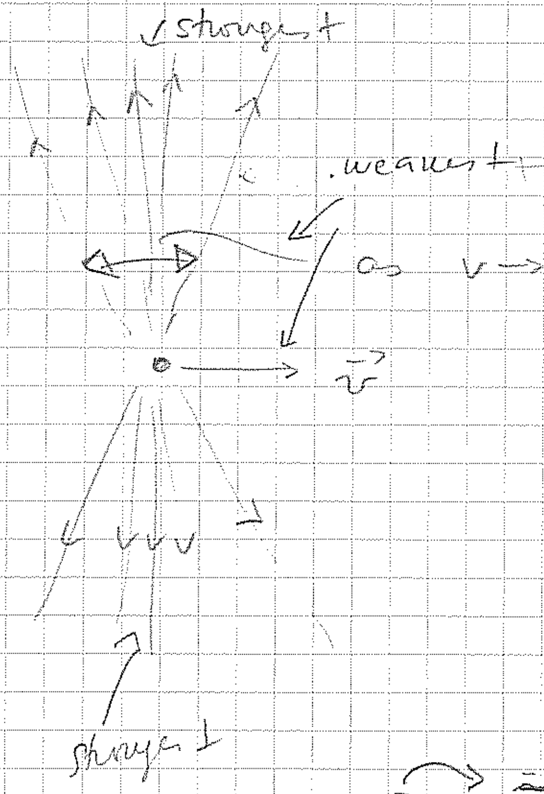
$$= \frac{v^2x^2+\cancel{v^4t^2}-2v^2\cancel{\bar{x}\cdot\bar{v}}\,t-(\bar{x}\cdot\bar{v})^2-\cancel{v^4t^2}+2v^2\cancel{t\,\bar{x}\cdot\bar{v}}}{|x-\bar{v}t|^2\,v^2}$$

$$= \frac{v^2x^2-(\bar{x}\cdot\bar{v})^2}{\;}$$ ∎

from $A^0 = \frac{e}{R}\frac{1}{\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}$, you can clearly see that $(A^0, \vec{A})$ is strongest @ $\sin^2\theta = 1$, i.e. $\theta = \pm\frac{\pi}{2}$ & weakest @ $\sin\theta \approx 0$, i.e. forward & backward from current position of charge.

The $\vec{E}$ field is (HW5)

$$\vec{E} = \frac{e\vec{R}}{R^3}\frac{1-v^2/c^2}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \quad \& \quad \vec{B} = \frac{1}{c}\vec{v}\times\vec{E}$$



as $v \to c$ denominator is small for $\theta \sim \frac{\pi}{2}$:

compare, for given $|\vec{R}|$,

$$\frac{|\vec{E}(\theta=\frac{\pi}{2})| - |\vec{E}(\theta=\frac{\pi}{2}+\Delta\theta)|}{|\vec{E}(\theta=\frac{\pi}{2})|} = \frac{\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} - \frac{1}{\left(1-\frac{v^2}{c^2}(1-(\Delta\theta)^2)\right)^{3/2}}}{\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}} \simeq$$

$$\left[\sin^2\left(\frac{\pi}{2}+\Delta\theta\right) \approx \left(1-\frac{(\Delta\theta)^2}{2}\right)^2 \approx 1-(\Delta\theta)^2\right]$$

$$\simeq 1 - \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1-\frac{v^2}{c^2}+\frac{v^2}{c^2}(\Delta\theta)^2\right)^{3/2}} \simeq 1 - \frac{1}{\left[1+\frac{v^2/c^2}{1-v^2/c^2}(\Delta\theta)^2\right]^{3/2}}, \text{ so,}$$

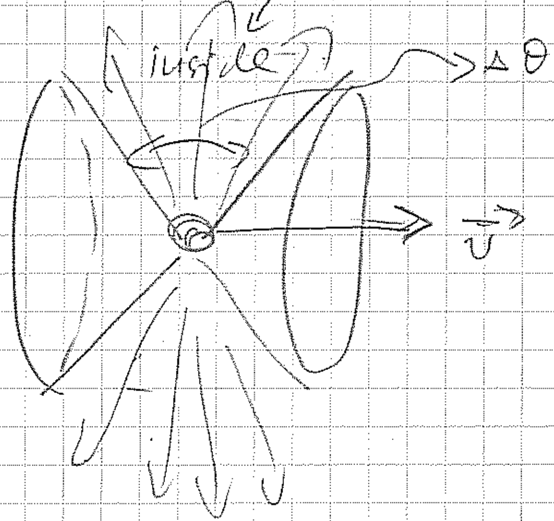
provided $\Delta\theta \lesssim \sqrt{1-\frac{v^2}{c^2}}$, 2nd term is $\ll$ 1st, so $|\vec{E}(\theta=\frac{\pi}{2})| - |\vec{E}(\frac{\pi}{2}+\Delta\theta)| \approx$

$\approx |\vec{E}(\theta = \frac{\pi}{2})|$; unreal — at $v/c \to 1$

we have appreciably $\neq 0$ EM fields only

in a narrow region

inside $\to \Delta\theta \sim \sqrt{1 - \frac{v^2}{c^2}}$



Back to our general expression — for a particle moving on arbitrary trajectory —

p. 148, note that $\vec{E}$ has two terms.

$$\vec{E} = e \frac{|\vec{x} - \vec{x}_c^{\,r}|}{\left((\vec{x} - \vec{x}_c^{\,r}) \cdot \vec{u}\right)^3} \left(c^2 - |\vec{v}_c^{\,r}|\right) \vec{u}$$

this term $\sim \frac{1}{R^2}$ as $R \to \infty$

much as for static charge

this term $\sim \frac{1}{R}$ as $R \to \infty$

$$+ e \frac{|\vec{x} - \vec{x}_c^{\,r}|}{\left((\vec{x} - \vec{x}_c^{\,r}) \cdot \vec{u}\right)^3} (\vec{x} - \vec{x}_c^{\,r}) \times (\vec{u} \times \dot{\vec{v}}_c^{\,r})$$

w/ $\vec{u} = c \frac{\vec{x} - \vec{x}_c^{\,r}}{|\vec{x} - \vec{x}_c^{\,r}|} - \vec{v}_c^{\,r}$; far away from charge, as $|\vec{x} - \vec{x}_c^{\,r}| = \vec{R} \to \infty$ we have $\vec{u} \to$ const so in 1st term, we have a contribution to $\vec{S} \sim \frac{1}{R^4}$ $\left(\vec{S} \sim \vec{E} \times \vec{B},\ \vec{E} \sim \frac{1}{R^2},\ \vec{B} \sim \frac{1}{R^2} \right)$

so $\oint_{S^2 \text{ of radius } R} d^2\vec{\sigma}\, \vec{S}$ due to 1st term $\to 0$ as $R \to \infty$

$$\Big(\sim \underbrace{R^2}_{S^2 \text{ area}} \times \frac{1}{R^4} \sim \frac{1}{R^2} \Big)$$

but 2nd term, $|\vec{E}| \sim \frac{1}{R}$, $|\vec{B}| \sim \frac{1}{R}$

gives $\vec{S} \sim \frac{1}{R^2}$ &

$$\underbrace{\oint_{S^2} d^2\vec{\sigma} \cdot \vec{S} \sim R^2 \frac{1}{R^2}}$$ $= \text{const}$ as $R \to \infty$

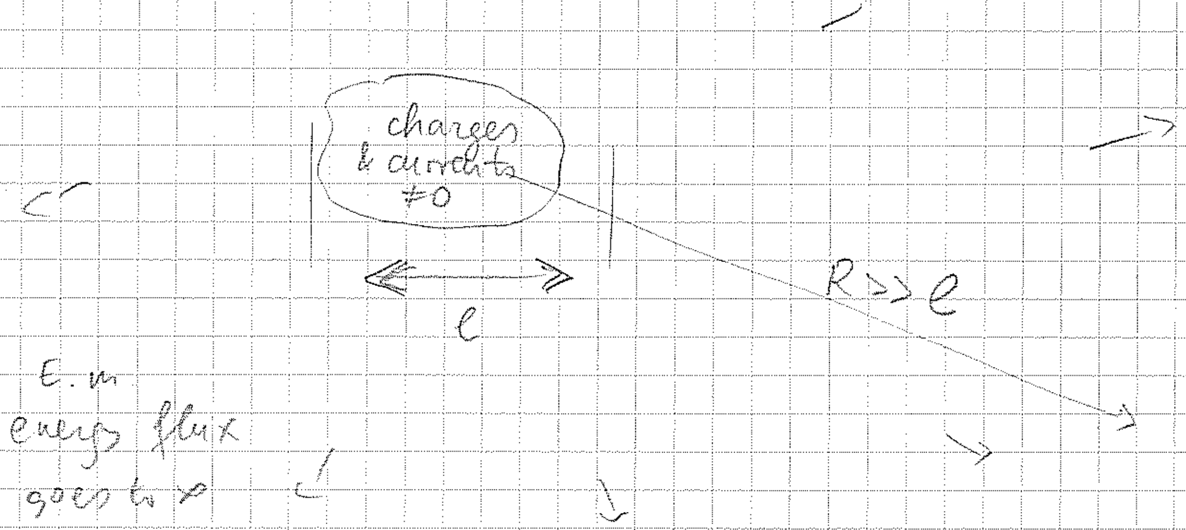
this 2nd term, $\sim \ddot{\vec{x}}_c$, the acceleration of the particle, is the cause of $\neq 0$ energy flux @ infinity.

$\hookrightarrow$ "el.m radiation carrying away energy"

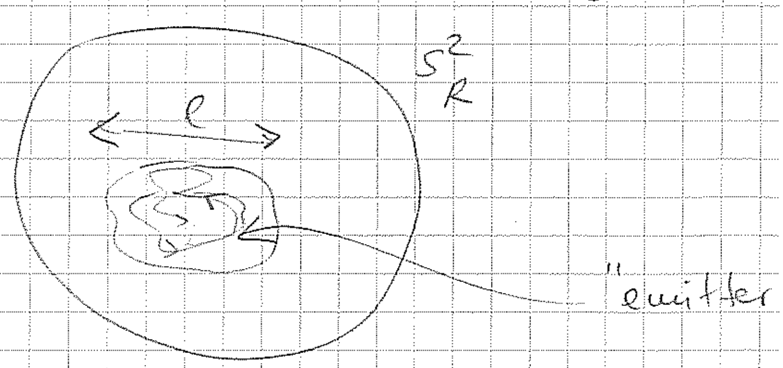
<u>Radiation</u> — as we saw, as $R \to \infty$,

(assuming charges & currents are in a bounded region)

we have $|\vec{E}| \sim \frac{1}{R}$ & $|\vec{S}| \sim \frac{1}{R^2}$



take a S^2_R - sphere of radius $R \gg \ell$ surrounding charges & currents



$$P(R) \equiv \oint_{S^2_R} \vec{S} \cdot d\vec{\sigma} =$$

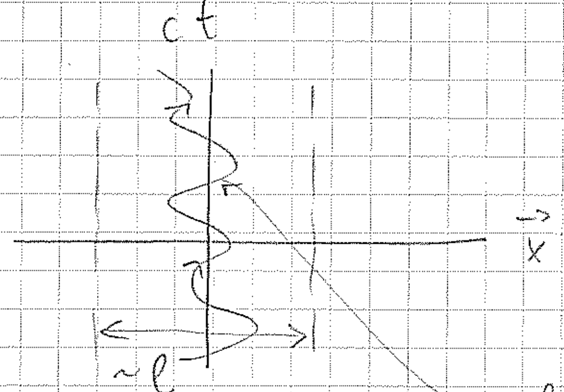
$$= \frac{\text{energy thru } S^2_R}{\text{unit time}} = \text{power thru } S^2_R$$

$$\lim_{R \to \infty} P(R) \equiv P_{\text{radiation}} \equiv \text{radiated power}$$

We have that

$$A^i(\bar{x},t) = \frac{1}{c}\int d^3\bar{x}' \underbrace{j^i\left(x', t - \frac{|\bar{x}-\bar{x}'|}{c}\right)} \frac{1}{|\bar{x}-\bar{x}'|} \qquad (155.1)$$

now these are assumed to be $\neq 0$ only in a bounded domain



worldline of particles inside $|\vec{x}| < l$ only (they are timelike, never mind my picture)

So $\int d^3\bar{x}'$ is limited to this $|\bar{x}'| \lesssim O(l)$ region

We'll be interested in the fields far away, @ $|\vec{x}| > l$, i.e. we'll expand

$$\frac{1}{|\bar{x}-\bar{x}'|} = \frac{1}{\sqrt{x^2 + x'^2 - 2\bar{x}\cdot\bar{x}'}} = \frac{1}{|x|}\frac{1}{\left(1 + \frac{|\vec{x}'|^2}{|\vec{x}|^2} - \frac{2\bar{x}\cdot\bar{x}'}{|x|^2}\right)^{1/2}}$$

$$\approx \frac{1}{|\vec{x}|}\left(1 + \frac{\bar{x}\cdot\bar{x}'}{|x|^2} + \dots\right)$$

leading $|\bar{x}|$-dependence at $|\bar{x}| \gg l \gtrsim |\bar{x}'|$

leading $\overbrace{\text{term}}^{\text{correction}}$ at $|\vec{x}| \gg l \gtrsim |\bar{x}'|$

(N.B.: we put origin of coordinate system somewhere in charge distribution, as in picture)

Now, in (155.1) $|\bar{x}-\bar{x}'|$ occurs at one other place $\longrightarrow$

→ the argument of the 4-current j^i:

$$j^i\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

Here, we'll again use
$$\frac{|\vec{x}-\vec{x}'|}{c} \approx \frac{1}{c}|\vec{x}|\left(1 - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \ldots\right)$$
$$= \frac{|\vec{x}|}{c} - \frac{\vec{x}\cdot\vec{x}'}{c|\vec{x}|} + \ldots$$

higher orders in $\frac{|\vec{x}'|}{|\vec{x}|} \lesssim \frac{\ell}{|\vec{x}|}$

& substitute into j^i:

$$j^i\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \approx j^i\left(\vec{x}', \underbrace{t - \frac{|\vec{x}|}{c}} + \frac{1}{c}\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|} + \ldots\right) \approx (*)$$

- this represents the leading retardation effect – field @ t @ $|\vec{x}| \gg \ell$ depends on currents at $t - \frac{|\vec{x}|}{c}$
- for convenience, denote $t_0 \equiv t - \frac{|\vec{x}|}{c}$

$$(*) = j^i\left(\vec{x}', t_0 + \underbrace{\frac{1}{c}\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|}} + \ldots\right)$$

this quantity is of order at most $\frac{\ell}{c}$, whereas $\frac{|\vec{x}|}{c} \gg \frac{\ell}{c}$

$$\approx j^i(\vec{x}', t_0) + \dot{j}^i(\vec{x}', t_0)\frac{\vec{x}\cdot\vec{x}'}{c|\vec{x}|} + \ddot{j}^i(\vec{x}', t_0)\,O\left(\left(\frac{\vec{x}\cdot\vec{x}'}{c|\vec{x}|}\right)^2\right) + \ldots$$

To have the expansion on the last line make sense, let's imagine that there is a typical frequency of motion of the charges and currents, i.e. $\dot{j}^i(\vec{x}\,', t_0) \sim \omega_0 j^i(\vec{x}\,', t_0)$

$$\left(\text{i.e. } \frac{\partial}{\partial t} \sim \omega_0\right)$$

Then ratio of 2nd term / 1st term is of order

$\dfrac{\omega_0 \vec{x}\cdot\vec{x}\,'}{c|\vec{x}|} \lesssim \dfrac{\omega_0 l}{c}$ — only if this is $\ll 1$ can we use this expansion. But $\dfrac{\omega_0}{c} \sim \dfrac{1}{\text{wavelength}} \sim \dfrac{1}{\lambda}$

So, our expansion requires that $\dfrac{l}{\lambda} \ll 1$

or $\lambda \gg l$

λ: wavelength of radiation

l: size of "emitter" (size of region where charges/currents that cause radiation are localized)

Hence: We're using 2 expansions: (1) $|\vec{x}| \gg l$

(2) $\lambda \gg l$.

— accept that & continue →

Back to (55.1), w/ $i=0$

$$A^0(\bar{x},t) = \frac{1}{c}\int d^3x' \, \frac{j^0(\vec{x}', t - \frac{|\bar{x}-\bar{x}'|}{c})}{|\bar{x}-\bar{x}'|} \approx$$

$$\approx \frac{1}{c}\int d^3\vec{x}' \, \frac{1}{|\vec{x}|}\left(1 + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + \ldots\right)\left(j^0(\vec{x}', t_0) + \frac{\vec{x}\cdot\vec{x}'}{c|\vec{x}|}\dot{j}^0(\vec{x}', t_0) + \ldots\right)$$

$$= \frac{1}{c}\frac{1}{|\vec{x}|}\overbrace{\int d^3x' \, j^0(\vec{x}', t_0)}^{\equiv cQ(t_0) = \text{charge of "emitter" @ } t_0}$$

$$+ \frac{1}{c}\frac{1}{|\vec{x}|^3}\,\vec{x}\cdot\overbrace{\int d^3x' \, \vec{x}' j^0(\vec{x}', t_0)}^{c\vec{d}(t_0) = \text{dipole moment of "emitter" @ } t_0}$$

$$+ \frac{1}{c^2}\frac{1}{|\vec{x}|}\frac{\vec{x}}{|\vec{x}|}\cdot\frac{d}{dt}\overbrace{\int d^3x' \, \vec{x}' j^0(\vec{x}', t_0)}^{c\vec{d}(t_0)} \qquad \left(\text{recall } t_0 = t - \frac{|\bar{x}|}{c}\right)$$

or
$$A^0(\bar{x},t) = \frac{1}{|\vec{x}|}Q\left(t - \frac{|\bar{x}|}{c}\right) + \frac{\vec{x}\cdot\vec{d}\left(t - \frac{|\bar{x}|}{c}\right)}{|\bar{x}|^3} + $$

$$+ \frac{\vec{x}\cdot\dot{\vec{d}}(t - |\bar{x}|/c)}{c|\bar{x}|^2} + \ldots$$

As for $\vec{A}(\bar{x},t)$, <u>only keep 1st term for now</u> (so replace $j^0 \to \vec{j}$)

$$A^\alpha(\bar{x},t) \simeq \frac{1}{c|\bar{x}|}\int d^3\vec{x}' \, j^\alpha(\vec{x}', t_0) \equiv \frac{1}{c|\vec{x}|}\int d^3x' (\vec{\nabla}_{x'} x'^\alpha)\cdot\vec{j}(\bar{x}', t_0) =$$

$$= \frac{1}{c|\bar{x}|}\int d^3x' \left[\vec{\nabla}_{x'}\left(x'^\alpha \vec{j}(\vec{x}', t_0)\right) - x'^\alpha \vec{\nabla}_{x'}\cdot\vec{j}(\vec{x}', t)\right] = \longrightarrow$$

$$= \frac{1}{c|\vec{x}|}\int d^3x'\ \vec{\nabla}_{x'}\left(x'^{\alpha}\,\vec{j}(\vec{x}',t_0)\right)$$

$\to$ a surface $\int$: $\left(\oint_{S^2_\infty} x'^{\alpha}\, d^2\vec{\sigma}\cdot\vec{j}(\vec{x}',t_0)\right)$ $\overset{\shortparallel}{0}$ @ $\vec{x}'\to\infty$

$$- \frac{1}{c|\vec{x}|}\int d^3x'\ x'^{\alpha}\underbrace{\left(\vec{\nabla}_{x'}\cdot\vec{j}(\vec{x}',t_0)\right)}_{-\frac{\partial}{\partial t}\rho(\vec{x}',t_0)}$$

by continuity equation

$$= \frac{1}{c}\frac{1}{|\vec{x}|}\int d^3x'\ x'^{\alpha}\dot{\rho}(\vec{x}',t_0) = A^{\alpha}(\vec{x},t)$$

$$\Rightarrow \quad \vec{A}(\vec{x},t) = \frac{1}{|\vec{x}|}\frac{1}{c}\frac{\partial}{\partial t}\int d^3x'\ x'^{\alpha}\,\rho(\vec{x}',t_0)$$

$$= \frac{1}{|\vec{x}|}\frac{1}{c}\frac{\partial}{\partial t}\vec{d}\left(t-\frac{|\vec{x}|}{c}\right)$$

So our formulae are

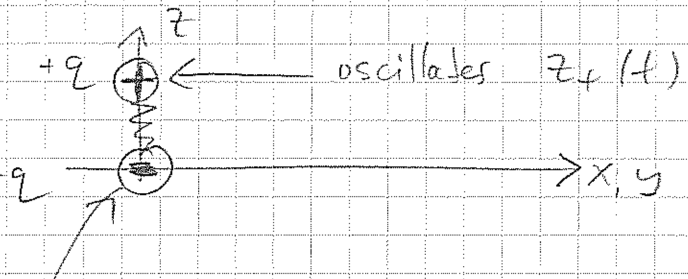
$$\begin{cases} A^0(\vec{x},t) = \dfrac{1}{|\vec{x}|}Q\left(t-\dfrac{|\vec{x}|}{c}\right) + \dfrac{\vec{x}\cdot\vec{d}\left(t-\frac{|\vec{x}|}{c}\right)}{|\vec{x}|^3} + \dfrac{\vec{x}\cdot\dot{\vec{d}}\left(t-\frac{|\vec{x}|}{c}\right)}{c\,|\vec{x}|^2} + \cdots \\[2ex] \vec{A}(\vec{x},t) = \dfrac{1}{|\vec{x}|}\dfrac{1}{c}\dot{\vec{d}}\left(t-\dfrac{|\vec{x}|}{c}\right) + \cdots \end{cases}$$

[[these terms may be worked out as needed – e.g. magnetic dipole radiation ...]]

We're interested in radiation, so let's consider some examples. First & foremost –

– <u>electric dipole radiation</u> – quite ubiquitous & relevant — e.g. atom "shaken" by EM wave radiates like an electric dipole

Model:



oscillates $z_+(t) = z_0 + a\sin\omega t$

– charge is fixed at $z_-(t) = 0$

hence dipole moment is $\vec{d} = \vec{e}_z\, q\,(z_0 + a\sin\omega t)$

(Note this allows us to not use $\rho(\bar{x},t)$, $\vec{j}(\bar{x},t)$, since we expressed A^0, $\vec{A}$ thru $\vec{d}$ & Q only. – here $Q=0$, hence we only have $\vec{d}$ to consider)

$$A^0(\bar{x},t) = \frac{\vec{x}\cdot\vec{d}\left(t-\frac{|\bar{x}|}{c}\right)}{|\bar{x}|^3} + \frac{\vec{x}\cdot\dot{\vec{d}}\left(t-\frac{|\bar{x}|}{c}\right)}{c\,|\vec{x}|^2} + \ldots$$

$$\vec{A}(\bar{x},t) = \frac{1}{|\bar{x}|c}\,\dot{\vec{d}}\left(t-\frac{|\bar{x}|}{c}\right) + \ldots \qquad \text{plug } \vec{d}(t) = q\vec{e}_z(z_0 + a\sin\omega t),$$

we have $\left(t_0 = t - \frac{|\bar{x}|}{c}\right)$

(160.1)
$$A^0(\bar{x},t) = \frac{z\,q\,(z_0 + a\sin(\omega t_0))}{|\vec{x}|^3} + \frac{z}{|\bar{x}|^2}\frac{1}{c}\,a\omega q\cos(\omega t_0) =$$

$$= q z_0 \frac{z}{|\vec{x}|^3} + q z_0 a \frac{\cancel{z}\sin(\omega t_0)}{|\vec{x}|^3} + \frac{z a\omega q}{|\bar{x}|^2 c}\cos(\omega t_0)$$

$$\vec{A}(\bar{x},t) = \frac{1}{|\bar{x}|c}\,\vec{e}_z\, q\, a\omega\cos(\omega t_0)$$

Before we continue, note that $\lambda \gg l$ means, for this case, that, since $l \sim z_0 + a$, $\underbrace{l\omega \ll c}$ $\left(\lambda \sim \frac{\omega}{c}\right)$ $\boxed{(v \sim l\omega \ll c)}$

$\boxed{\text{charge is moving NONRELATIVISTICALLY}}$

Next, we want to extract the $\sim \frac{1}{|\vec{x}|}$ pieces of $\vec{E}, \vec{B}$ from (160.1)

i.e. the radiation fields

Since $\cos/\sin\left(\omega\left(t-\frac{|\vec{x}|}{c}\right)\right)$ are always bounded, they don't die off w/ $|\vec{x}|$, hence only the 3^{rd} term in A^0 — the one that came from $\underline{\frac{\vec{x}\cdot\dot{\vec{d}}(t_0)}{c|\vec{x}|^2}}$ contributes to the radiation field

$$\left(\text{recall}\quad \vec{E} = -\vec{\nabla} A^0 - \frac{\partial}{\partial(ct)}\vec{A} \qquad \vec{B} = \vec{\nabla}\times\vec{A}\right)$$

if $\vec{\nabla}_x$ acts on $\frac{z}{|\vec{x}|^2}$ produces $O\left(\frac{1}{|x|^2}\right)$ so neglect

Hence, we have:

$$\vec{E} = -\vec{\nabla}_x A^0 - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} = -\vec{\nabla}_x\left(\frac{z a\omega q}{\vec{x}^2 c}\cos\left(\omega\left(t-\frac{|\vec{x}|}{c}\right)\right)\right)$$

$$-\frac{1}{c^2}\frac{1}{|\vec{x}|}\vec{e}_z\, a\omega q\frac{\partial}{\partial t}\cos\left(\omega\left(t-\frac{|\vec{x}|}{c}\right)\right) =$$

$$= +\frac{z a\omega q}{\vec{x}^2 c}\,\omega\sin\left(\omega\left(t-\frac{|\vec{x}|}{c}\right)\right)\left(-\frac{1}{c}\vec{\nabla}|\vec{x}|\right)$$

$$+\frac{1}{c^2}\frac{\vec{e}_z}{|\vec{x}|}a\omega q\,\omega\sin\left(\omega\left(t-\frac{|\vec{x}|}{c}\right)\right) =$$

$$= \frac{a\omega^2 q}{c^2}\sin\left(\omega\left(t-\frac{|\vec{x}|}{c}\right)\right)\frac{1}{|\vec{x}|}\left(-\frac{\vec{x}}{|\vec{x}|}\frac{z}{|\vec{x}|}+\vec{e}_z\right)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\frac{\vec{e}_z q a \omega}{|\vec{x}| c} \cos \omega t_0 \right) = (*)$$

if $\vec{\nabla}$ acts here, produces non-radiation field, hence only act on $\cos(\omega t_0)$

$$(*) - \frac{q a \omega \vec{e}_z}{|\vec{x}| c} \times \vec{\nabla} \cos(\omega t_0) = - \frac{q a \omega \vec{e}_z}{|\vec{x}| c} \left(-\omega \sin(\omega t_0)\right) \vec{\nabla} t_0$$

$$= + \frac{q a \omega^2}{|\vec{x}| c} \sin(\omega t_0) \left(-\frac{1}{c}\right) \left(\vec{e}_z \times \underbrace{\vec{\nabla} |\vec{x}|}_{\frac{\vec{x}}{|\vec{x}|}}\right) =$$

$$\left\{ \begin{aligned} \vec{B} &= - \frac{q a \omega^2}{c^2 |\vec{x}|} \sin(\omega t_0)\, \vec{e}_z \times \frac{\vec{x}}{|\vec{x}|} , \quad \text{while from bottom of (16} \\ \vec{E} &= \frac{q a \omega^2}{c^2 |\vec{x}|} \sin(\omega t_0) \left(\vec{e}_z - \frac{\vec{x}}{|\vec{x}|} \frac{z}{|\vec{x}|} \right) \end{aligned} \right\} (162.1)$$

Note that $\vec{B} = \frac{\vec{x}}{|\vec{x}|} \times \vec{E}$ — proof:

$$\frac{\vec{x}}{|\vec{x}|} \times \vec{E} = \underbrace{\frac{q a \omega^2}{c^2 |\vec{x}|} \sin \omega t_0}_{= -\ddot{\vec{d}}(t_0)} \left(\frac{\vec{x}}{|\vec{x}|} \times \vec{e}_z - \cancel{\frac{\vec{x}}{|\vec{x}|} \times \frac{\vec{x}}{|\vec{x}|} \frac{z}{|\vec{x}|}} \right) \quad \blacksquare$$

in short, nonrelativistic dipole has rad'n field

$$\boxed{\vec{E}(\vec{x},t) = \frac{1}{c^2 |\vec{x}|} \left[-\ddot{\vec{d}}(t_0) + \hat{\vec{r}} \left(\ddot{\vec{d}}(t_0) \cdot \hat{\vec{r}} \right) \right] ; \quad \ddot{\vec{d}} = \vec{e}_z (-q a \omega^2 \sin \omega t_0) \quad \& \quad \vec{B} = \hat{\vec{r}} \times \vec{E}}$$

this has an even more succinct expression

-- use $(\vec{B} \times \hat{\vec{r}}) \times \hat{\vec{r}} = -\vec{B} + (\hat{\vec{r}} \cdot \vec{B})\,\hat{\vec{r}}$ (***)

Proof: $\epsilon^{ijk} (B \times \hat{r})^j \hat{r}^k =$

$= \epsilon^{ijk} \epsilon^{jlm} B^l r^m r^k$

$= (-\delta^{il}\delta^{km} + \delta^{im}\delta^{lk}) B^l \hat{r}^m \hat{r}^k = -B^i + \hat{r}^i (\vec{B} \cdot \hat{r})$

$$\vec{E} = \frac{1}{c^2 |\vec{x}|} (\ddot{\vec{d}} \times \hat{\vec{r}}) \times \hat{\vec{r}}$$

$$\vec{B} = \hat{\vec{r}} \times \vec{E}$$

most elegant form for the radiation from a nonrelativistic dipole

$\sim \ddot{\vec{d}} \equiv$ acceleration needed to have radiation (we already knew that, of course)

next: what's the physics of this —
= power, etc. — ?
= angular distrib'n ?

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \text{use formulae (162.1)} =$$

$$= \frac{c}{4\pi}(-)\left(\frac{q a \omega^2 \sin(\omega t_0)}{c^2 |\vec{x}|}\right)^2 \Big(\underbrace{\vec{e}_z \times (\vec{e}_z \times \hat{\vec{r}})}_{\underbrace{-(\vec{e}_z \times \hat{r}) \times \vec{e}_z}_{+(\hat{\vec{r}} \times \hat{\vec{e}}_z) \times \hat{\vec{e}}_z}} - \underbrace{\frac{z}{|\vec{x}|} \hat{\vec{r}} \times (\vec{e}_z \times \hat{\vec{r}})}_{+\frac{z}{|\vec{x}|} (\vec{e}_z \times \hat{r}) \times \hat{r}} \Big)$$

use (***) $\longrightarrow$

$$-\hat{\vec{r}} + \underbrace{(\hat{\vec{r}} \cdot \vec{e}_z)}_{\frac{z}{|\vec{x}|}} \vec{e}_z + \cancel{\frac{z}{|\vec{x}|}(-\vec{e}_z)} + \underbrace{\frac{z}{|\vec{x}|}}_{(\hat{\vec{r}} \cdot \vec{e}_z)} (\hat{\vec{r}} \cdot \vec{e}_z)\, \hat{\vec{r}}$$

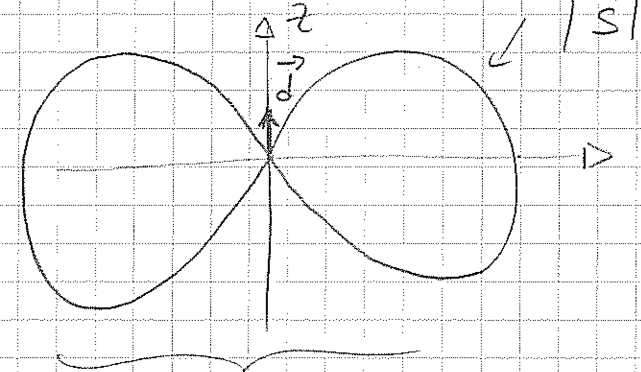
here

$$\vec{S} = \frac{c}{4\pi} \frac{|\ddot{\vec{d}}(t_0)|^2}{c^4 |\vec{x}|^2} \hat{\vec{r}} \underbrace{\Big(1 - \underbrace{(\hat{\vec{r}} \cdot \vec{e}_z)^2}_{\cos^2\theta}\Big)}_{\sin^2\theta}$$

[this of course applies when $\vec{d} \parallel \vec{e}_z$]

$$\vec{S} = \hat{\vec{r}} \, \frac{\sin^2\theta \, |\ddot{\vec{d}}(t_0)|^2}{4\pi c^3 |\vec{x}|^2}$$

$|\vec{S}| = \text{const}$, where $|\sin\theta| = |\vec{x}| \times (\text{const.})$



<u>NB:</u> this is a typical directional characteristic of most antennas (as simplest ones are dipoles)

to calculate power radiated,

$$P(R) = \oint_{S^2_R} d^2\vec{\sigma} \cdot \vec{S} = \oint_{S^2_R} d^2\vec{\sigma} \cdot \hat{\vec{r}} \; \frac{\sin^2\theta \, (qa\omega^2 \sin\omega t_0)^2}{4\pi c^3 R^2} = \boxed{***}$$

$\sin^2 \omega t_0 = \sin^2\!\left(\omega\left(t - \frac{R}{c}\right)\right)$, comes out of angular $\int$

$$\int d^2\vec{\sigma} \cdot \hat{\vec{r}} = d^2 A = 2\pi R^2 \int_0^1 d\theta \, \sin\theta \quad \text{(integrated over } \varphi)$$

$$P(R)\Big|_{R\to\infty} = \frac{1}{2}\frac{1}{c^3}\underbrace{\int_0^\pi d\theta \sin^3\theta}\ \ddot{\vec{d}}^{\,2}(t_0)$$

$$\int_0^\pi d\theta\sin^3\theta = -\int_0^\pi d(\cos\theta)(1-\cos^2\theta) = \int_{-1}^{1} d\mu(1-\mu^2) = \left(\mu-\frac{\mu^3}{3}\right)\Big|_{-1}^{1} =$$

$$= 1-\frac{1}{3}-\left(-1+\frac{1}{3}\right) =$$

$$= \frac{2}{3}+\frac{2}{3} = \frac{4}{3}$$

$$P = \frac{2}{3c^3}\ddot{\vec{d}}^{\,2}$$

power radiated by dipole

now since $\ddot{\vec{d}}^{\,2} = q^2 a^2 \omega^4 \sin^2(\omega t_0)$

if we avge per period $\sin^2\omega t_0 \to 1/2$

$$P = \frac{q^2 a^2}{3c^3}\omega^4 \quad \longrightarrow \quad E, B \sim \text{acceleration} \sim \omega^2, \quad \text{power} \sim \omega^4$$

power radiated per unit time
(averaged over period of dipole oscillation)

power is stronger @ higher frequencies

why the sky is blue ?

($\lambda_{blue} < \lambda_{red}$ $\omega_{blue} > \omega_{red}$

...