In the HW, you will study energy-momentum conservation w/ matter and will show that when energy & momentum of matter are included, the conservation laws still hold...

— " — hw 6, Problem 1.

If we look at our expression for T^{jk} from p 177,

we can see that if $\vec{E}$ & $\vec{B}$ are parallel (at the $(\vec{x}, t)$ we're looking at, then, taking $\vec{E} \parallel \vec{B}$ along $\hat{x}$, say

$$T^{ij} = \left(\begin{array}{c|c|c|c} \frac{E_x^2 + B_x^2}{8\pi} & 0 & 0 & 0 \\ \hline & -\frac{1}{8\pi}(E_x^2 + B_x^2) & 0 & 0 \\ \hline & & +\frac{1}{8\pi}(E_x^2 + B_x^2) & 0 \\ \hline & & & +\frac{1}{8\pi}(E_x^2 + B_x^2) \end{array} \right)$$

i.e. $T^{00} = \mathcal{E}$ = energy density

$T^{11} = -\mathcal{E}$

$T^{22} = \mathcal{E}$

$T^{33} = \mathcal{E}$.

So, at a given $(\vec{x}, t)$ T^{ij} can be diagonalized — meaning, that a frame exists, where it's diagonal, whenever one can find a frame where $\vec{E} \parallel \vec{B}$. $\rightarrow$ we know that this can always be done unless $\vec{E} \perp \vec{B}$ & $|\vec{E}| = |\vec{B}|$ *

(including if one is $= 0$)

The final questions I'd like to discuss are about the electromagnetic energy of a system of charges, the "friction" due to radiation, and the limitations of classical electrodynamics.

(not placed in an external field – but charges create their

let's start from a simple case – consider a (closed) system of charges, which are all nonrelativistic in the chosen frame of reference. This means that the leading effects of these charges will be the creation of electrostatic fields and their mutual Coulomb interactions.

– recall source in Maxwell's equations is

$$\frac{4\pi}{c} j^{\ell} = \left(4\pi\rho, \frac{4\pi}{c}\vec{j}\right) =$$

$$= \left(4\pi\rho, \frac{4\pi}{c}\vec{v}\rho\right)$$

$\uparrow$ creation of $\vec{A}$ & hence $\vec{B}$ field is an $\underline{O}\left(\frac{v}{c}\right)$ effect

- also recall that power radiated by an electric dipole (a nonrelativistic one) is Power $= \frac{2}{3c^3} \ddot{\vec{d}}^{\,2}$

$$\dot{\vec{d}} \sim \vec{v}$$

$$\ddot{\vec{d}} = \frac{d}{dt} \vec{v}$$

hence is also suppressed by powers of $\left(\frac{v}{c}\right)^2$ -- & more -- later --

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Since all particles are nonrelativistic, we have, omitting self-energies $\sum_a (-m_a c^2)$ in Lagrangian

recall $S = \sum_a \int_{\substack{\text{worldline} \\ \text{(of a-th particle)}}} (-m_a c) ds = \sum_a -m_a c^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{\vec{v}_a^{\,2}}{c^2}}\, dt$

action for particles.

$$\approx \sum_a -m_a c^2 (t_2 - t_1) + \frac{1}{2} \sum_a \int_{t_1}^{t_2} dt\, \frac{m_a \vec{v}_a^{\,2}}{2}$$

nonrelativistic limit

so $\underline{L = \sum_a \frac{1}{2} m_a \vec{v}_a^{\,2}}$

nonrelativistic L for the system of $a = 1 \ldots N$ free particles

Next, include the fact that particles are charged and create EM fields --

-- nominally, for every particle, we'd have to include

$$\underbrace{L^{\text{int.}}}_{} = \sum_a \left(\cancel{q_a \frac{\vec{v}_a}{c} \cdot \vec{A}(\vec{x}_a, t)} - q_a \varphi(x_a, t) \right)$$

(interaction between particles & EM field)

let's neglect this in the leading $\frac{|\vec{v}|}{c} \ll 1$ approximation

so we have, to leading order in v/c, the Lagrangian

$$L = \sum_a \left(\frac{m_a \vec{v}_a^{\,2}}{2} - q_a \varphi(\vec{x}_a, t) \right)$$

Now what is $\varphi(\vec{x}_a, t)$ here?

Since we are working in the limit $\frac{v}{c} \to 0$, we are completely ignorant of retardation effects, so we don't have to include the EM field as separate dynamical d.o.f. (it does not propagate in the $c \to \infty$ limit).

The way we'll go about determining $\varphi(x_a, t)$ is by taking $\varphi(x_a)$ to be the electrostatic potential created by all other particles $b \neq a$ at x_a. It will have no explicit dependence on t, but will depend only on the instantaneous (& simultaneous) positions of all other charges. To justify this, let's ask what is the energy of the EM field created by a system of static charges?

$\downarrow (v/c) \to 0$ limit

(so they'll only create Coulomb)

We know
$$\mathcal{E} = \frac{1}{8\pi} \int_{V \text{ all space}} d^3x \left(\vec{E}^2 + \underbrace{\vec{B}^2}_{0} \right) =$$

$$= \frac{1}{8\pi} \int_V d^3x \; \vec{E} \cdot (-\vec{\nabla}\phi) = -\frac{1}{8\pi} \int_V d^3x \left(\vec{\nabla}(\vec{E}\phi) - \phi\, \vec{\nabla}\cdot\vec{E} \right)$$

$$= -\frac{1}{8\pi} \underbrace{\oint_{\partial V = S^2_\infty} d^2\vec{\sigma} \cdot \vec{E}\phi} + \frac{1}{8\pi} \int d^3x \; \phi \underbrace{(\vec{\nabla}\cdot\vec{E})}_{4\pi\rho(\vec{x})} = (*)$$

by Maxwell's eqn

assume charges are localized –
– so $\vec{E} \to \phi$ $\vec{x} \to \infty$
– thus $\oint_{S^2_\infty} = 0$.

$$(*) \Rightarrow \quad \mathcal{E} = \frac{1}{2} \int d^3x \; \phi(\vec{x}) \rho(\vec{x})$$

what's $\rho(\vec{x})$? – $\rho(\vec{x}) = \sum_a q_a \delta^{(3)}(\vec{x} - \vec{x}_a)$ –

the charge density of our system of charges $\longrightarrow$

hence

$$\mathcal{E} = \sum_a \frac{1}{2} q_a \phi(\vec{x}_a) = (**)$$

the Electrostatic energy of a system of static (or $v/c \ll 1$) charges

value of scalar potential @ position of a-th charge: of course, this is determined by solving

$$\vec{\nabla} \cdot \vec{E} = 4\pi \underbrace{\sum_a q_a \delta^{(3)}(\vec{x} - \vec{x}_a)}_{\rho(\vec{x})}$$

$$\Vert$$

$$-\vec{\nabla} \cdot \vec{\nabla} \phi = 4\pi \sum_{a'} q_{a'} \delta^{(3)}(\vec{x} - \vec{x}_{a'})$$

$$\Delta \phi(\vec{x}) = -4\pi \sum_{a'} q_{a'} \delta^{(3)}(\vec{x} - \vec{x}_{a'})$$

$\Downarrow$ hence

$$\boxed{\phi(\vec{x}) = \sum_a \frac{q_a}{|\vec{x} - \vec{x}_a|}}$$

(**) so

$$\mathcal{E} = \sum_a \frac{1}{2} q_a \times \sum_{a'} \frac{q_{a'}}{|\vec{x}_a - \vec{x}_{a'}|}$$

$$\mathcal{E} = \frac{1}{2} \sum_a q_a \sum_{a'} \frac{q_{a'}}{|\vec{x}_a - \vec{x}_{a'}|}$$

this has an unpleasant feature. at $a = a'$ this is infinite. $\Rightarrow$ simply because the electrostatic energy of a charge is ∞ – and it is, because the electrostatic field of a charge diverges at the position of the charge. — should we be bothered?

this ∞ happens, 'cause charge is considered pointlike ---

-- if it had size $\frac{1}{r_e}$, we'd have (take $q = e$)

that $\underbrace{\frac{e^2}{r_e}}_{\text{order of magnitude of "size" of electron}} \sim \underbrace{m_e c^2}_{\text{such that electrostatic energy} \sim \text{rest energy}}$ when $r_e \sim \frac{e}{m_e c^2}$

$\boxed{r_e \sim \frac{e}{m_e c^2} \Longleftrightarrow \text{classical radius of the electron}}$

(if we just consider $a = a' = 1$, we'd have indeed $\mathcal{E} = \frac{1}{2} \frac{q_a^2}{r_e}$)

(classical EM forces upon us to consider non-pointlike electron -- - but r_e is tiny $\sim 10^{-13}$ cm) classical EM breaks down @ much larger distances

In practice, we say this ① ②

$$\mathcal{E} = \frac{1}{2} \sum_{aa'} \frac{q_a q_{a'}}{\bar{X}_a - \bar{X}_{a'}} = \underbrace{\frac{1}{2} \sum_{a=a'} \frac{q_a^2}{|X_a - X_a|^2}}_{} + \underbrace{\frac{1}{2} \sum_{a \neq a'} \frac{q_a q_{a'}}{|X_a - X'_a|}}_{\text{this is interaction energy of charges}}$$

- this is an ∞ constant $= \Sigma$ of "self energies" of charges
- independent of separations between charges

So we split $\mathcal{E}$ in two parts –

1 – a Σ of self energies, independent of separations between charges

$\underbrace{}_{\text{formally infinite}}$

hence, this term will be dropped – at relative distances $\gg 10^{-13}$ cm & @ $v/c \ll 1$ can consider charges as pointlike.

- ⊛ this will not affect how charges move around relative to each other
- ⊛ it may affect their (internal) structure – but this involves distance scales that we do not probe (10^{-13} cm) – and at these scales classical EM does not apply
- ⊛ it will be absorbed in rest mass "renormalization": $m_e = (m_0 + EM) = $ (measured value) (both m_0 & EM self energy are formally ∞: reflect our ignorance of structure)

2 – an interaction term that depends on distances between charges & affects their relative motion. ⟸ keep. affects how charges behave.

they may have structure – but unless we can study it, it is irrelevant for physics at long distances – which is all we attempt to describe anyway (e.g. what is seen). We can fantasize about the structure at $\Delta r \ll 10^{-13}$ cm – but unless we can probe this experimentally this remains irrelevant fantasy.

Upshot:

2 only.

We let $\mathcal{E} = \frac{1}{2} \sum_{a \neq a'} \frac{q_a q_{a'}}{|\bar{x}_a - \bar{x}_{a'}|}$

$= \sum_{a<a'} \frac{q_a q_{a'}}{|\bar{x}_a - \bar{x}_{a'}|}$

electrostatic energy of a system of charge

This comes from:

Lagrangian of a system of charges (m_a, q_a) in $v_a \ll c$ limit.

(189.1) $$L^{(0)} = \sum_a m_a \frac{\vec{v}_a^{\,2}}{2} - \sum_{a<b} \frac{q_a q_b}{|\bar{x}_a - \bar{x}_b|}$$

- to leave behind any doubt - this describes (microscopic) of most of physics we see — solid state, liquids, gases, galaxies . . . (we use it in NR QM, too - $L \to H$)
- it's got its limitations - but so does physics in general — we have no good knowledge of $\Delta r \to 0, \Delta t \to 0$ or $\Delta r, \Delta t \to \infty$, for that matter — . . .
- magnetic effects are not included - but we can do so to next order in v/c.

We can go now to next order in $\frac{v}{c}$ expansion...

— eqn. (85.1) called $L^{(0)}$ — 0-th order in $\frac{v}{c}$

— we can go to $\left(\frac{v}{c}\right)^2$, actually

(— radiation $\sim \frac{1}{c^3}$ — so occurs in next order only —)

We start w/ Lagrangian for a-th particle

$$L_a = -m_a c^2 \sqrt{1 - \frac{v_a^2}{c^2}} - q_a \phi + \frac{q_a}{c} \vec{v}_a \cdot \vec{A} -$$

— & calculate ϕ & $\vec{A}$ at $\vec{x}_a$ induced by all other charges — if we get ϕ to $\underline{O}\left(\frac{v^2}{c^2}\right)$ & $\vec{A}$ to $\underline{O}\left(\frac{v}{c}\right)$, we'll have L_a (when we plug them back) to $\underline{O}\left(\frac{v^2}{c^2}\right)$...

We use the general expressions for retarded ϕ & $\vec{A}$

$$\phi(\vec{x}, t) = \int d^3\vec{x}' \, \frac{\rho(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c})}{|\vec{x} - \vec{x}'|}$$

since charges are nonrelativistic ρ will not change much for times $\sim \frac{|\vec{x} - \vec{x}'|}{c}$, so we can expand

we want this @ $\vec{x}_a$

density of all other charges

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expanding ρ in $\frac{|\bar{x}-\bar{x}'|}{c}$

this is total charge of system - won't change in time $\to 0$.

we have

$$\phi(\bar{x},t) = \int d^3x' \frac{\rho(\bar{x}',t)}{|\bar{x}-\bar{x}'|} - \frac{\partial}{\partial t}\left(\int d^3x' \frac{\rho(\bar{x}',t)}{\cancel{|\bar{x}-\bar{x}'|}}\right)\frac{\cancel{|\bar{x}-\bar{x}'|}}{c}$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial t^2}\int d^3x' \frac{\rho(\bar{x}',t)}{\cancel{|\bar{x}-\bar{x}'|}} \frac{|x-x'|^2}{c^2} + \dots$$

$$\phi(\bar{x},t) \approx \int d^3x' \frac{\rho(\bar{x}',t)}{|\bar{x}-\bar{x}'|} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\left(\int d^3x' |\bar{x}-\bar{x}'| \rho(\bar{x}',t)\right)$$

(191.1)

as for $\bar{A}$, we do the same:

$$\vec{A}(\bar{x},t) = \frac{1}{c}\int \frac{\vec{J}\left(\bar{x}', t - \frac{|\bar{x}-\bar{x}'|}{c}\right)}{|\bar{x}-\bar{x}'|} d^3x' =$$

$$\approx \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}', t)}{|\bar{x}-\bar{x}'|}$$

to leading order in $\vec{v}/c$ ($\vec{A}$ has an $O(\frac{v}{c})$ term in L already.)

(191.2)

to make 191.1 & 191.2 manageable, imagine ρ correspond to a single charge at $\bar{x}_0$ $\longrightarrow$

this means $\rho(\vec{x}', t) = q\, \delta^{(3)}(\vec{x}' - \vec{x}_0(t))$

$$\vec{j}(\vec{x}', t) = q\, \dot{\vec{x}}_0(t)\, \delta^{(3)}(\vec{x}' - \vec{x}_0(t))$$

$$= q\, \vec{v}_0(t)\, \delta^{(3)}(\vec{x}' - \vec{x}_0(t))$$

hence:

$$\underbrace{\varphi(\bar{x}, t)}_{(191.1)} = \frac{q}{|\vec{x} - \vec{x}_0(t)|} + \frac{q}{2c^2} \underbrace{\frac{\partial}{\partial t^2}}_{\partial_t \left(\frac{\partial}{\partial t} \cdots\right)} |\vec{x} - \vec{x}_0(t)|$$

(191.2)

$$\hookrightarrow \quad \vec{A}(\vec{x}, t) \approx \frac{q}{c} \frac{\vec{v}_0(t)}{|\bar{x} - \bar{x}_0(t)|}$$

Trick: let

(for convenience, really!!)

$$\phi'(\bar{x}, t) = \phi(\bar{x}, t) - \frac{1}{c} \frac{\partial f(\bar{x}, t)}{\partial t}$$

$$\vec{A}'(\bar{x}, t) = \vec{A}(\bar{x}, t) + \bar{\nabla} f(\bar{x}, t)$$

w/ $$f = \frac{q}{2c} \frac{\partial}{\partial t} |\vec{x} - \vec{x}_0(t)|$$

$$\Longrightarrow \quad \phi'(\bar{x}, t) = \frac{q}{|\bar{x} - \bar{x}_0(t)|}$$

$$\vec{A}'(\bar{x}, t) = \frac{q\, \vec{v}_0(t)}{|\bar{x} - \bar{x}_0(t)|} + \frac{q}{2c} \vec{\nabla} \frac{\partial}{\partial t} |\vec{x} - \vec{x}_0(t)|$$

$$= \frac{q\, \vec{v}_0(t)}{|\bar{x} - \bar{x}_0(t)|} + \frac{q}{2c} \frac{\partial}{\partial t} \underbrace{\vec{\nabla} |\vec{x} - \vec{x}_0(t)|}_{(\bar{x} - \bar{x}_0(t))/|\bar{x} - \bar{x}_0(t)| \,\equiv\, \vec{n}(t)}$$

$$\vec{A}'(\bar{x},t) = \frac{q\,\bar{v}_0(t)}{|\bar{x}-\bar{x}_0(t)|} + \frac{q}{2c}\,\dot{\vec{n}}(t)$$

$$\vec{n}(t) = \frac{\bar{x}-\bar{x}_0(t)}{|\bar{x}-\bar{x}_0(t)|}$$ unit vector from $\bar{x}_0(t)$ to $\bar{x}$

$$\dot{\vec{n}}(t) = -\frac{\dot{\vec{x}}_0(t)}{|\bar{x}-\bar{x}_0(t)|} - \frac{\bar{x}-\bar{x}_0(t)}{|\bar{x}-\bar{x}_0(t)|^2}\,\frac{d}{dt}|\bar{x}-\bar{x}_0(t)| =$$

$$= -\frac{\vec{v}_0(t)}{|\bar{x}-\bar{x}_0(t)|} - \frac{\vec{x}-\vec{x}_0(t)}{|\bar{x}-\bar{x}_0(t)|}\,\frac{(\bar{x}-\bar{x}_0(t))\cdot(-\vec{v}_0(t))}{|\bar{x}-\bar{x}_0(t)|^2}$$

$$\Rightarrow \dot{\vec{n}}(t) = \frac{-\vec{v}_0(t) + \vec{n}(t)\,(\vec{n}\cdot\vec{v}_0(t))}{|\bar{x}-\bar{x}_0(t)|}$$

for many charges - $\sum$ their effects

So:
$$\begin{cases} \phi'(\bar{x},t) = \dfrac{q}{|\bar{x}-\bar{x}_0(t)|} \\[2ex] \vec{A}'(\bar{x},t) = \dfrac{q\,\vec{v}_0(t) + \vec{n}(t)(\vec{n}(t)\cdot\bar{v}_0(t))\,q}{2c\,|\bar{x}-\bar{x}_0(t)|} \end{cases}$$

in
$$L_a = \underbrace{-m_a c^2\sqrt{1-\frac{v_a^2}{c^2}}} - q_a\phi' + \frac{q_a}{c}\,\bar{v}_a\cdot\vec{A}'$$

$$-m_a c^2 + \frac{1}{2}m_a v_a^2 + \frac{1}{8}\frac{m_a v_a^4}{c^2}$$

$$L_a = \frac{m_a \vec{v}_a^{\,2}}{2} + \frac{1}{8}\frac{m_a \vec{v}_a^{\,4}}{c^2} - q_a \sum_{b \neq a} \frac{q_b}{r_{ab}}$$

$$+ \frac{q_a}{c} \vec{v}_a \cdot \sum_{b \neq a} \frac{q_b \vec{v}_b + \vec{n}_{ba}(\vec{n}_{ba} \cdot \vec{v}_b) q_b}{2c\, r_{ab}}$$

$$L_a = \frac{m_a v_a^2}{2} + \frac{1}{8}\frac{m_a v_a^4}{c^2} - q_a \sum_{b \neq a} \frac{q_b}{r_{ab}}$$

$$+ \frac{q_a}{2c^2} \sum_{b \neq a} \frac{q_b\, \vec{v}_a \cdot \vec{v}_b + (\vec{v}_a \cdot \vec{n}_{ba})(\vec{v}_b \cdot \vec{n}_{ba})}{r_{ab}}$$

Lagrangian of a-th particle - moving in the field of the other particles ($\sum_{b \neq a}$) - to order $\frac{v^2}{c^2}$!!

So magnetic interactions are also included, to leading $(v/c)^2$ order

What is L for the entire system ??

principle ?
= guidance - should yield same EOM for a-th particle as L_a does!

check this! e.g. in electrostatic term

- to achieve this last two terms should be $\sum$ over NOT $\sum_b \sum_{\substack{a \\ a \neq b}}$ but $\sum_{b > a}$.

$$L = \sum_a \left(\frac{m_a \vec{v}_a^{\,2}}{2} + \frac{1}{8} \frac{m_a (\vec{v}_a^{\,2})^2}{c^2} \right)$$

$$- \sum_{a<b} \frac{q_a q_b}{r_{ab}} + \sum_{a<b} \frac{q_a q_b}{2c^2 r_{ab}} \left(\bar{v}_a \cdot \bar{v}_b + (\bar{v}_a \cdot \bar{n}_{ab})(\bar{v}_a \cdot \bar{n}_{ab} \right.$$

(Darwin, 1922)
(gradson of ...)
also Charles

H is best found by $L_0 + \delta L \to H_0 + \delta H = H_0 - \delta L$

w/ $v_a = \frac{p_a}{m}$

$$H = \sum_a \frac{p_a^2}{2m_a} - \frac{p_a^4}{8c^2 m_a^3} + \sum_{a>b} \frac{q_a q_b}{r_{ab}}$$

"Darwin Hamiltonian"

$$- \sum_{a>b} \frac{q_a q_b}{2c^2 m_a m_b r_{ab}} \left(\bar{p}_a \cdot \bar{p}_b - (\bar{p}_a \cdot \bar{n}_{ab})(\bar{p}_b \cdot \bar{n}_{ab}) \right)$$

<u>useful</u> → e.g. in many-body quantum!!
(or classical)

→ in atomic physics - corrections to spectra due to v^2/c^2

→ plasmas

all good

↳ <u>However</u> at $\underline{\mathcal{O}\left(\frac{v}{c}\right)^3}$ <u>must</u> include effects of radiation - & the d.o.f of the EM field as dynamical!