$\| I_{ij} \| \equiv$ rank-2 3-tensor

$\downarrow$ 2 indices; $\downarrow$ $i,j = 1,2,3$

under rotations, every index gets transformed as a 3-vector, e.g.

$$I'_{ij} = \sum_{l,m=1}^{3} O_{il}\, O_{jm}\, I_{lm}$$

where O_{il} is a 3×3 orthogonal matrix ($\in SO(3)$)

With higher rank tensors, the story is same.

Similarly, one can introduce 4-tensors of rank 2, 3, ...

(a 4 vector x^i is a "4-tensor of rank 1")

We'll do this as we go along and we need the necessary math! $\Rightarrow$ so we'll occasionally have to (come back to end of Ch. 1...)

But it is important to recall the basics:

$$X' = \hat{O} X \quad \text{or} \quad X^{i\prime} = \sum_{j=0}^{3} O^{ij} x^j$$

$$\hat{O} \in SO(1,3)$$

where

$$\| O^{ij} \| = \gamma \begin{pmatrix} 1 & +v/c & 0 & 0 \\ +v/c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\in SO(1,3)$? e.g.

$$X^T G X = X^{T\prime} G' X' = X^{T\prime} G X'$$

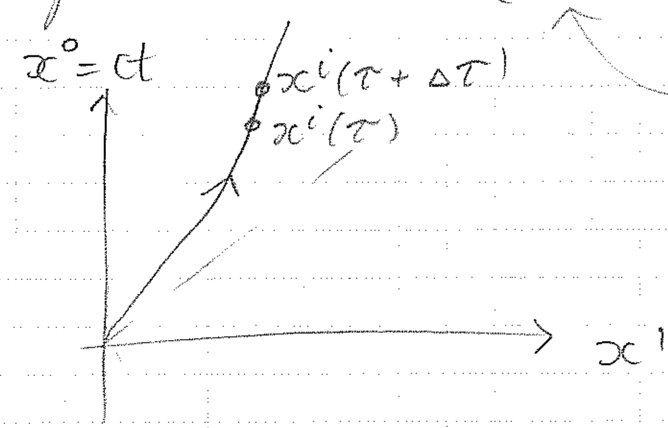
$$x'_i x^{i\prime} = x^{i\prime} g_{ij} x^{j\prime} = x_i x^i$$

Let's not forget why we introduced 4-vectors — in order to easily keep track of how quantities transform between different inertial frames. Ultimately, when we write down Eq. of motion (or actions), we'll demand that they be the same in all frames, when expressed in the variables appropriate to each frame. This means that the LHS & RHS of the Eq. of motion should be expressed as four vectors and thus the relation between the two sides will hold in any frame.

(just like $m\ddot{\vec{r}} = \vec{f}$ holds in any rotated frame in nonrelativistic physics)

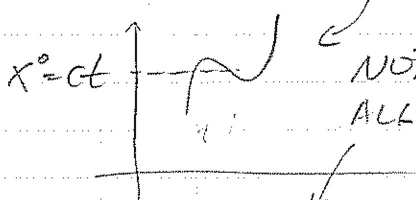
So we need to learn how to write down EOM in 4-vector notation.

Let x^i be the 4 vector describing position (in space time) of a particle. The worldline of a particle is some timelike curve in spacetime $x^i(\tau)$.



parameter describing where on the worldline the particle is; in principle τ can be taken to be any parameter, in particular, one can take $\tau = ct$ where t is the time coordinate in some inertial frame.

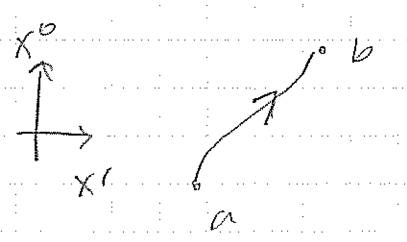
(because worldline is always timelike



where two points on curve would have same t)

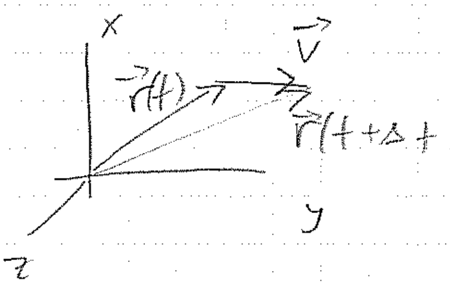
But, it is particularly useful to take τ to be the proper time, i.e. the "spacetime length" of the worldline – recall

$$\tau_{ab} = \int_a^b \frac{ds}{c}$$



the reason is that this τ is a Lorentz invariant measure of length (while ct is NOT!), i.e. it is a Lorentz "scalar".

Then, in analogy w/ usual velocity



$$\vec V = \left.\frac{\vec r(t+\Delta t) - \vec r(t)}{\Delta t}\right|_{\Delta t \to 0} = \frac{d\vec r}{dt}$$

we can define a "4 velocity":

4 vector

$$u^i = \left.\frac{x^i(\tau+\Delta\tau) - x^i(\tau)}{c\,\Delta\tau}\right|_{\Delta\tau\to 0} = \frac{dx^i}{ds}$$

scalar !!

4-scalar

to make it dim-less

4 vector

In the inertial frame shown on top of (47)

$$u^i = \frac{dx^i}{ds} = \frac{dx^i}{c\,dt\sqrt{1-\left(\frac{d\vec x}{dt}\right)^2\frac{1}{c}}}\,, \quad \text{since } x^i = (ct, \vec x)$$

$$u^0 = \frac{d(ct)}{c\,dt\sqrt{1-\vec V^2/c^2}} = \frac{1}{\sqrt{1-\vec V^2/c^2}} = \gamma \qquad \vec u = \frac{d\vec x/dt}{c\sqrt{1-\vec V^2/c^2}} = \gamma\frac{\vec V}{c}$$

So 4 velocity

$$u^i = \left(\gamma, \gamma \frac{\vec{v}}{c}\right)$$

The "length" or "norm" of u^i is

$$u^i g_{ij} u^j = u^{0\,2} - \vec{u}^{\,2} =$$

$$= \gamma^2 - \gamma^2 \frac{\vec{v}^{\,2}}{c^2} = \frac{1 - \vec{v}^{\,2}/c^2}{1 - \vec{v}^{\,2}/c^2} = \underline{1}.$$

The "4-acceleration" is, clearly,

$$\frac{du^i}{ds} = w^i \quad , \quad \text{in analogy w/} \quad \frac{d\vec{v}}{dt} ,$$

and is always "$\perp$" to 4-velocity:

Since $u^i u_i = 1$

$$\frac{d}{ds}(u^i u_i) = 0$$

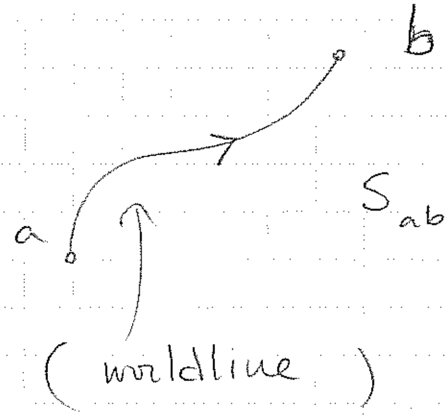
$$\Rightarrow \quad \frac{d}{ds} u^i \, u_i + u^i \frac{du_i}{ds} = 0$$

$$\parallel$$

$$\frac{du_i}{ds} u^i + u^i \frac{du_i}{ds} = 2 \frac{du_i}{ds} u^i = 2 w_i u^i$$

Finally, we're ready for some dynamics...

First, need free-particle action:



$S_{ab} = ?$ => should be Lorentz invariant

(since E.O.M. must be same in all frames, and they're obtained by extremizing S_{ab})

The length of the world line in spacetime is the only invariant characteristic —

– so $S_{ab} = \text{const} \underbrace{\int_a^b ds}$

this is max. for straight line among all worldlines (see earlier notes ≈ p. 24 or so)

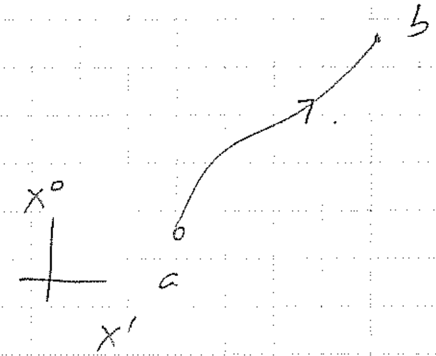
hence const = $-\alpha$, to get min for straight line

Now $[s]$ = length while $[S_{ab}]$ = energy × time = = momentum × distance

Hence const = $-\alpha$ = – [something w/ dimension of momentum, which is a Lorentz scaler]

call this $\underline{mc}$

So -- $S_{ab} = -mc \int_a^b ds$



- action for relativistic "point particle"
- depends on c & m.

c: Lorentz scalar

m: Lorentz scalar, dim $\equiv$ kg ("dim of mass")

Many things we can do w/ S_{ab} -- to follow..

for now: recall

$$ds = c\,dt\sqrt{1 - \vec{v}^2/c^2} \quad \text{i.t.o. } x^0 - x^1 \text{ inertial coordinates}$$

so $$S = -mc^2 \int_{t_a}^{t_b} dt \underbrace{\sqrt{1 - \vec{v}^2/c^2}}_{\text{if } |\vec{v}| \ll c}$$

$$\simeq -mc^2(t_b - t_a) + mc^2 \int_{t_a}^{t_b} \frac{1}{2}\frac{\vec{v}^2}{c^2}\,dt + O\left(\left(\frac{v}{c}\right)^4\right)$$

so nonrelat. limit sensible!

$$= -mc^2(t_b - t_a) + \int_{t_a}^{t_b} dt\, \frac{m\vec{v}^2}{2} + \cdots$$

(-- $m \equiv$ mass --)

$-mc^2(t_b - t_a)$: does not affect variational problem.

$\frac{m\vec{v}^2}{2}$: nonrelativistic kinetic energy = free particle $L = \frac{m\vec{v}^2}{2}$!