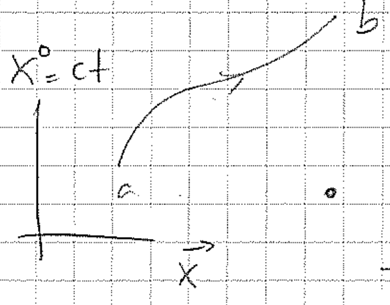
As usual S is useful to
(a) find EOM
(b) find conserved quantities (symmetries →
→ conserv. laws)

(a) get EOM ---



$$S_{ab} = -mc \int_a^b ds$$

- let's work EOM in a given inertial frame. – where $ds = c\,dt \sqrt{1 - \vec{v}^2/c^2}$

- we'll try to write answer in 4-vector notation – and if we can, this means these EOMs are true in any frame

So
$$S_{ab}[\vec{x}] \equiv -mc \int_{t_a}^{t_b} dt \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2 \frac{1}{c^2}}$$

every time like trajectory is specified by giving $\vec{x}(t)$

this is action evaluated on trajectory $\vec{x}(t)$

need $\delta S = S[\vec{x} + \delta\vec{x}] - S[\vec{x}]$ (to linear order in $\delta\vec{x}$) $= 0$

this is just like any old nonrelativistic problem, except

$$L(\vec{x}, \dot{\vec{x}}, t) = -mc \sqrt{1 - \left(\frac{d\vec{x}}{dt}\right)^2 \frac{1}{c^2}}$$

we know EOM's are

$$\frac{\partial L}{\partial \vec{x}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}}$$

$$\overset{\shortparallel}{0} = \frac{d}{dt}\frac{\partial}{\partial \dot{\vec{x}}}\left(-mc\sqrt{1-\dot{\vec{x}}^2/c^2}\right)$$

So E.O.M. is: $\frac{d}{dt}\frac{d}{d\dot{\vec{x}}}\sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}} = 0$

$$\frac{d}{dt}\left(\frac{\dot{\vec{x}}/c}{\sqrt{1-\dot{\vec{x}}^2/c^2}}\right) = 0$$

$$\frac{d}{dt}(\gamma\vec{v}) = 0$$

$$\frac{d}{dt}(u^{1,2,3}) = 0$$

since $u^i = (\gamma, \gamma\vec{v})$

Now this is NOT YET a 4-vector equation — but it implies one w/ some work:

$$0 = \frac{d}{dt}(\gamma\vec{v}) = \frac{d}{dt}\left[\left(1-\frac{\vec{v}^2}{c^2}\right)^{-1/2}\vec{v}\right] = \left(1-\frac{v^2}{c^2}\right)^{-1/2}\left[\dot{\vec{v}} + \frac{\vec{v}\,(\vec{v}\cdot\dot{\vec{v}})}{c^2-v^2}\right] = 0$$

Clearly $\dot{\vec{v}} = 0$ is a solution $\Rightarrow \vec{v} = \text{const} \Rightarrow \vec{x} = \vec{v}t + \vec{x}_0$

But is there a solution w/ $\dot{\vec{v}} \neq 0$? → Claim: NO: if $\dot{\vec{v}} \neq 0$ then use $\dot{\vec{v}} + \frac{\vec{v}(\vec{v}\cdot\dot{\vec{v}})}{c^2-v^2} = 0$, $\times\ \vec{v} \Rightarrow \vec{v}\cdot\dot{\vec{v}} + \frac{v^2}{c^2-v^2}\vec{v}\cdot\dot{\vec{v}} = 0 \Rightarrow$

$\Rightarrow \vec{v}\cdot\dot{\vec{v}}\frac{c^2}{c^2-v^2} = 0 \Rightarrow \vec{v}\cdot\dot{\vec{v}} = 0 \Rightarrow$ but then $\dot{\vec{v}} = 0$ — contradict #

So then $\frac{d}{dt}\gamma\vec{v} = 0 \Rightarrow \vec{v} = \text{const} \Rightarrow$

$$\Rightarrow \frac{d}{dt}\gamma = 0$$

but then also $\frac{du^0}{dt} = 0$ since $u^0 = \gamma$

so we have $\frac{du^i}{dt} = 0$ -- this is almost

a 4-vector relation $\rightarrow$ $\times$ by $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (never 0 or ∞

$$\frac{d\,u^i}{\sqrt{1-\frac{v^2}{c^2}}\,dt} = 0 \Rightarrow \underbrace{\frac{du^i}{ds} = 0}_{\text{"4 acceleration = 0"}}$$

$\Updownarrow$

4 vector EOM

Similarly we can find conserved quantities!

Symmetries: $\underbrace{\begin{aligned} t &\rightarrow t + \text{const} \\ \vec{x} &\rightarrow \vec{x} + \text{const} \end{aligned}}_{\text{homogeneity of space and time}}$

$$\underline{x^i \rightarrow x^i + \text{const}}$$

Again use $L = -mc\sqrt{1-\frac{\vec{v}^2}{c^2}}$

("generalized momentum")

homogeneity of space $\Rightarrow$ $\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = 0$

$$\frac{d}{dt}\left(mc\gamma\vec{v}\right) = 0$$

by what we already did (– used EOM)

$$\frac{d}{dt}\left(mcu^{1,2,3}\right) = 0$$

$$\frac{d}{dt}\left(mcu^{i}\right) = 0$$

$$\frac{d}{ds}\left(mcu^{i}\right) = 0$$

4 momentum conservation

$$p^i = mcu^i = \left(\gamma mc, mc\gamma\frac{\vec{v}}{c}\right) = \left(\gamma mc, \gamma m\vec{v}\right)$$

Recall for $L = L(q, \dot{q}, t)$

$$\frac{d}{dt}L = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \cancel{\frac{\partial L}{\partial t}}$$

assume no explicit t-dep.

EOM

$$= \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\dot{q}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right) \Rightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right)}_{\text{conserved quantity}} = 0$$

$\equiv E$ – energy

For our $L = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}}$

$$E = \vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L =$$

$$= \vec{v}\left(-mc^2\right)\frac{1}{2\sqrt{1-\vec{v}^2/c^2}}\left(-2\frac{\vec{v}}{c^2}\right) + mc^2\sqrt{1-\vec{v}^2/c^2}$$

$$= \frac{m\vec{v}^2}{\sqrt{1-\vec{v}^2/c^2}} + mc^2\sqrt{1-\vec{v}^2/c^2} =$$

$$= \frac{m\vec{v}^2 + mc^2\left(1-\vec{v}^2/c^2\right)}{\sqrt{1-\vec{v}^2/c^2}} = \frac{mc^2}{\sqrt{1-\vec{v}^2/c^2}}$$

So energy is $E = \gamma mc^2$ $\qquad \frac{d}{dt}E = 0$

& $\frac{d}{dt}\big(\underbrace{mcu^i}\big) = 0$ for free particle.

four-momentum

$$p^i = mcu^i \quad \text{conservation}$$

$$\boxed{p^i = (p^0, \vec{p}), \quad p^0 = \gamma mc = \frac{E}{c}, \quad p^ip_i = m^2c^2}$$

$$\boxed{\vec{p} = \gamma m\vec{v} = \frac{E\vec{v}}{c^2}}$$

Notice: as $\vec{v} \to 0$ $\quad E \to mc^2$ = "rest energy"

(necessary consequence of relativity)