<u>Note:</u> ① the only "mass" ever worth talking about is the "m" in $S = -mc\int ds$

It is a Lorentz invariant ("scalar", characteristic of any particle, i.e. the same in all frames

② the energy is $E = \gamma mc^2 = cp^0$

NB: (Σp^i) for a closed system is conserved – use in particle reactions, decays...

$\vec{p} = \gamma m \vec{v}$

$\left(p^0 = \frac{E}{c}, \vec{p} = \gamma m \vec{v}\right) \equiv p^i$: 4-vector of energy/momentum

$\underline{p^i p_i} = \frac{E^2}{c^2} - \vec{p}^{\,2} = c^2\gamma^2 m^2 - \gamma^2 m^2 \vec{v}^{\,2} = \underline{m^2c^2}$

"4-vector p^i of particle of mass m obeys $\underline{p^i p_i = m^2c^2}$

③ $m \to 0$? "massless particle"

$E = \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \longrightarrow 0$ unless $|\vec{v}| \to c$ when $E = \frac{0}{0}$

while $\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2 \to 0$

so: $\underbrace{E = |\vec{p}|c}$

$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \longrightarrow 0$ unless $|\vec{v}| \to c$ when $\vec{p} = \frac{0}{0}$

<u>for massless particles</u>

photon (only)

$\underline{p^{0\,2} - \vec{p}^{\,2} = 0} \Leftrightarrow p^i p_i = 0$

So we learned that

$$S_{ab} = -mc \int_a^b ds = -mc^2 \int_{t_a}^{t_b} dt \sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2}$$

$a = (ct_a, \vec{x}_a)$ where $\vec{x}(t_a) = \vec{x}_a$

$b = (ct_b, \vec{x}_b)$ $\vec{x}(t_b) = \vec{x}_b$

E.O.M is simply

$$\frac{du^i}{ds} = 0 \quad , \text{ or } \quad \frac{d}{dt}(\gamma\vec{v}) = 0$$

$$\left(\frac{d}{dt}\vec{v} = 0 \quad \& \quad \frac{d}{dt}\gamma = 0 \right)$$

conserved quantities
due to translations of x^i are $p^i = (p^0, \vec{p})$

$$p_0 = \frac{\mathcal{E}}{c} = mc\gamma$$

$$\& \quad \vec{p} = \gamma m \vec{v}$$

The nonrelativistic limit is

$$S = \underbrace{-mc^2(t_b - t_a) + \int dt \frac{1}{2} m \left(\frac{d\vec{x}}{dt}\right)^2}_{\substack{\text{nonrelativistic} \\ \text{free-particle Lagrangian}}} + \underbrace{\cdots}_{\left(\frac{v}{c}\right)^2 \text{ corrections}}$$

<u>Note:</u> we obtained the EOM & the conservation of p^i using a particular coordinate system where coordinates are $(t, \vec{x})$

Clearly, result is independent of the frame used — since $ds = c\,dt\sqrt{1-\frac{\vec{v}^2}{c^2}} = c\,dt'\sqrt{1-\frac{\vec{v}'^2}{c^2}}$

the derivation of EOM/conservation laws holds in any frame (primed as well as un-primed) — this is reflected in being able to write both EOM & conservation law i.t.o. 4-vectors:

$\frac{du^i}{ds} = 0$ & $p^i = \text{const}$ along trajectories of free particle

Both EOM & conservation laws can also be Directly obtained in 4-vector form. This will be useful when discussing "4-angular momentum" (quantities conserved due to rotations & boosts) & the properties of the E&M field (later).

If curious, read §10 (transforms of distribution functions)

& §14 (4-tensor of angular momentum)

on your own ...

We now want to consider more than free particles. In Newtonian mechanics, particles interact via potentials - which always are thought to result from either electromagnetic or gravitational interactions (i.e. are macroscopic consequences of E&M & gravity).

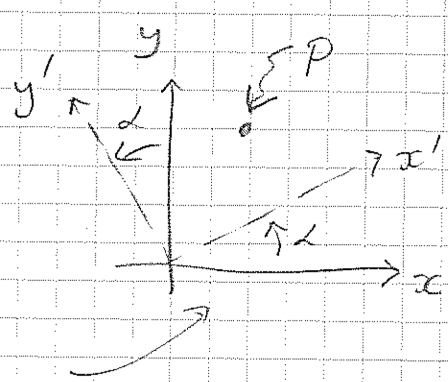
How would we describe this in a relativistic theory?

In terms of the action, we'd have to add something to the particle's action that would represent it's interaction w/ an "external potential"

now this potential can depend on $\vec{x}$ & t, of course, and it must have definite transformation properties under SO(1,3) Lorentz symmetry transforms

If we had no experiment at hand, simplest possibility would be to have a scalar potential. It's value $\forall\ \vec{x}, t$ would be given by $\varphi(\vec{x}, t)$ — a scalar "field" (w.r.t Lorentz)

We call it a "field" meaning it has values $V(\vec{x}, t)$. A "scalar" means this:



rotations of space

let $P = (x,y)$ or (x',y') in rotated frame

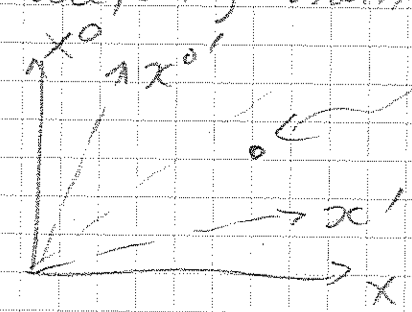
We say "φ is a scalar under rotations"

if $\varphi'(x',y') = \varphi(x,y)$

$$\underbrace{\begin{pmatrix}\text{value of field}\\ \text{at } P \text{ using}\\ \text{rotated}\\ \text{coordinate system}\end{pmatrix} = \begin{pmatrix}\text{value of field}\\ \text{at } P \text{ using}\\ \text{non-rotated}\\ \text{system}\end{pmatrix}}$$

this is familiar from E&M — "scalar potential" $\varphi(\vec{x},t)$ is an example of a scalar field under spatial rotations

Now, a scalar under Lorentz transforms would be something similar, but now we include boosts



$- \varphi(x^0, \vec{x}) = \varphi'(x^{0'}, \vec{x}')$

$$\begin{pmatrix}\text{value of}\\ \varphi \text{ @ } P\\ \text{in unprimed}\\ \text{frame}\end{pmatrix} = \begin{pmatrix}\text{value of}\\ \varphi \text{ at } P\\ \text{in primed}\\ \text{frame}\end{pmatrix}$$

A field for which $\varphi(x^i) = \varphi'(x'^i)$ is called a <u>Lorentz scalar field</u>

If we'd want to describe the action of a particle in an external $\varphi(x^0, \vec{x})$ - scalar we could simply write

$$S = \underbrace{-mc\int ds}_{\substack{S_{free} \\ \text{"relativistic kinetic term"}}} + \underbrace{\int ds\, \varphi(x^i)}_{\substack{S_{interaction} \\ \text{describes inter'n w/ } \varphi(x^0,\vec{x})}}$$

- S_{free} is a Lorentz scalar, as we know
- so is $S_{interaction} = \int ds\, \varphi(x^i)$ -
  - since ds is scalar
  - & $\varphi(x^i)$ is scalar, too

Problem is, such "scalar fields" that classical particles can propagate through & interact with have not been observed.

((this is not to say that <u>quantum</u> scalar fields do not exist — they do $\longrightarrow$

– for example $\underbrace{\pi^0, \pi^+, \pi^-, K^{0,\pm}}$ ...

these are all "scalar fields"
Lorentz

– but they do not correspond to
(a) stable quantum excitations, capable of making up a classical field to be studied by us in this class

and

(b) they mediate very short-range interactions (1 fm or so), so wouldn't be of macroscopic relevance even if stable.

Note also Higgs boson – if it exists – is supposed to be a scalar field – alas also not stable, not long-range ...))

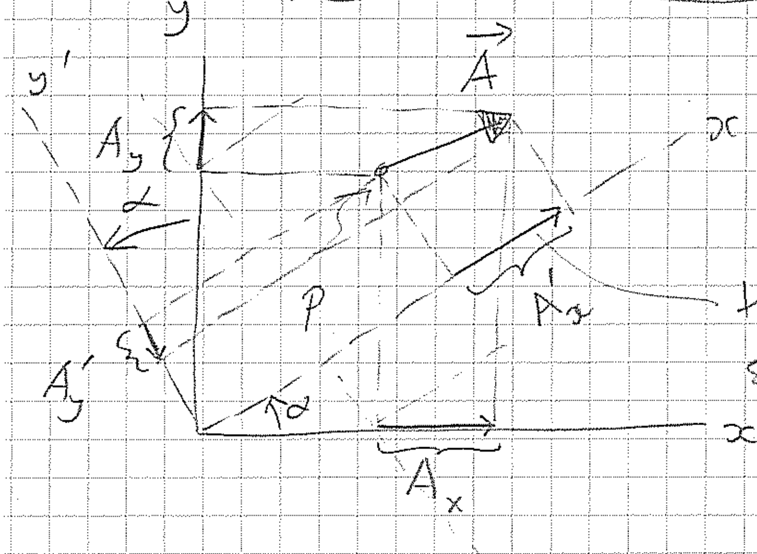
So we're forced – by experiment – to go to the "next-best" thing ... the next possibility is to have a field which is a Lorentz vector – "A^i"

main topic of this class

OK, but let's 1st remember what a vector under rotations is ---

$\underline{Ex}$: $\underbrace{\vec{A}(\vec{x})}_{}$: a 3-vector $\forall\ \vec{x}$

"3- vector field"



the arrow is put here only to signify that $A'_x \sim$ (difference between value of x' @ front & back end)

$$A'_x(x', y') = \cos\alpha\ A_x(x,y) + \sin\alpha\ A_y(x,y)$$

$$A'_y(x', y') = -\sin\alpha\ A_x(x,y) + \cos\alpha\ A_y(x,y)$$

so
$$\begin{pmatrix} A'_x(x',y') \\ A'_y(x',y') \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A_x(x,y) \\ A_y(x,y) \end{pmatrix}$$

or more generally
$$\underbrace{\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix}}_{\text{components of } \vec{A} \text{ in } (x'y'z')} (\vec{x}') = \hat{O} \underbrace{\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}}_{\text{components of } \vec{A} \text{ in } (x,y,z)} (\vec{x})$$

we call this "a 3-vector field $\vec{A}(\vec{x})$"

SO(3) matrix rotating $\vec{x}$ into $\vec{x}'$

Note that, of course $\vec{A}(\vec{x}) \cdot \vec{y}$, for any 3-vector $\vec{y}$ invariant under rotations, since it's a dot product, and any dot product is.

((So, we know that electrostatic potential $\varphi(\vec{x}, t)$ & vector potential $\vec{A}(\vec{x}, t)$ are a 3-scalar & a 3-vector... However, clearly, they can't form a scalar under $SO(1,3)$, because $\vec{A}$ is a 3-vector. So, the most natural option is to combine them into a Lorentz vector...))

Let's now define a 4-vector field $A^i(x^0, \vec{x})$, same as for 3-vector...

recall

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}' = \underbrace{\hat{O}}_{\text{an } SO(1,3) \text{ matrix (6 parameters, 3 boosts / 3 rotns)}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

So

$$\left.\begin{pmatrix} A^{0'} \\ A^{1'} \\ A^{2'} \\ A^{3'} \end{pmatrix}(x^{0'}, \vec{x}') = \hat{O} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}(x^0, \vec{x})\right\} \text{ a "4-vector field"}$$

in short, $A^i(\underset{\uparrow}{x})$

x means 4-vector denoting $(ct, \vec{x})$

$\underbrace{}$

a set of 4 quantities $\forall\,(ct, \vec{x})$, transforming as a 4-vector under Lorentz group.

So how will our particle interact w/ A^i?
need a 4-scalar to put in $S_{interaction}$.

Since A^i is a 4-vector, need another 4-vector – $\dfrac{dx^i}{ds} = u^i$.

So can write: $S_{int} = \text{const} \int ds \, \dfrac{dx^i}{ds} g_{ij} A^j(x)$

(integral over worldline ds – scalar

(particle's 4-velocity)

(value of A^j on worldline

$\underbrace{}_{\text{Lorentz scalar}}$

((Can imagine other terms...
e.g. $\int ds \, A^i g_{ij} A_j$ -- but not so good, not gauge invt...))

So we arrive at
action for a relativistic particle in an external 4-vector field

$$S_{charged} = -mc\int ds - \frac{e}{c}\int ds\, u^i A_i$$

We need now
<u>two</u> Lorentz scalar quantities to characterize (a) particle itself = m (<u>mass</u>)
(b) the "strength" of its interaction w/ A_i $\equiv e$ (<u>charge</u>)

(Since we have not chosen units of A^i yet, let's leave question of units of "e" yet; written const. as $\frac{e}{c}$ for convenience now...)

What is this S in more conventional notation → use $A^i = (\varphi, \vec{A})$ → $u^i A_i = u^0 A^0 - \vec{u}\cdot\vec{A}$

$$S = -mc^2\int dt\sqrt{1-\frac{\vec{v}^2}{c^2}} - \frac{e}{c}\int ds\left(u^0\varphi - \vec{u}\cdot\vec{A}\right) =$$

$$= -mc^2\int dt\sqrt{1-\vec{v}^2/c^2} - \frac{e}{c}\int c\,dt\left(\underbrace{\sqrt{1-\frac{\vec{v}^2}{c^2}}\,\gamma}_{1}\,\varphi - \underbrace{\sqrt{1-\frac{\vec{v}^2}{c^2}}\,\gamma}_{1}\,\frac{\vec{v}}{c}\cdot\vec{A}\right)$$

$$S = -mc^2\int dt\sqrt{1-\vec{v}^2/c^2} - e\int dt\left[\varphi(\vec{x},t) - \frac{\vec{v}}{c}\cdot\vec{A}(\vec{x},t)\right] \longrightarrow$$

Now this one is familiar from

HW 1 of Classical Mechanics (PHY 354)

We know that the EOM describe the

motion of a charged particle in φ & $\vec{A}$:

we have

$$S = \int dt \underbrace{\left[-mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} - e\,\varphi(\vec{x},t) + \frac{e}{c}\,\vec{v}\cdot\vec{A}(\vec{x},t) \right]}_{L(\vec{x},\vec{v},t)}$$

$L(\vec{x},\vec{v},t)$; to get Euler-Lagrange eqns:

$$\frac{\partial L}{\partial \vec{v}} = \frac{mc^2}{\sqrt{1-v^2/c^2}}\,\frac{\vec{v}}{c^2} + \frac{e}{c}\vec{A}(\vec{x},t)$$

Σ over $k=1,2,3$ implied

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = m\,\frac{d}{dt}(\gamma\vec{v}) + \frac{e}{c}\,\frac{\partial \vec{A}}{\partial x^k}\,v^k + \frac{e}{c}\,\frac{\partial \vec{A}}{\partial t}$$

$$\frac{\partial L}{\partial x^i} = -e\,\frac{\partial}{\partial x^i}\varphi + \frac{e}{c}\,\vec{v}\cdot\frac{\partial \vec{A}}{\partial x^i}$$

equating last two eqns (here $i=1,2,3$)

$$m\frac{d}{dt}(\gamma v^i) = e\underbrace{\left(-\frac{\partial}{\partial x^i}\varphi - \frac{1}{c}\frac{\partial A^i}{\partial t}\right)}_{\left(-\frac{\partial A^i}{c\,\partial t} - \frac{\partial \varphi}{\partial x^i}\right) = E^i} + \frac{e}{c}\underbrace{\left(v^j\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^k}v^k\right)}_{?}$$

let's say just a name for now.

From this page on, let us make the convention that $\underbrace{i, j}$ = 0,1,2,3 (4-vectors)

reserved for 3-vectors : $\underbrace{\mu, \nu, \ldots}$ = 1,2,3 (3 vectors)

While to "contract" i, j need g_{ij} (to obtain a Lorentz-invariant expression), for 3-vector indices "μ, ν" there's no such need: $\vec{x} \cdot \vec{y} = x^\mu y^\mu$

$\vec{A} \cdot \vec{x} = A^\mu x^\mu$

etc. . .

and for μ, ν, I will be sloppy regarding upper or lower indices !!

only when converting to 4-vectors one needs to remember that $A^{1,2,3} \to A^{i=1,2,3}$, & for x etc. –

upper

So, EOM is

$$m \frac{d}{dt}\left(\gamma v^\mu\right) = e \underbrace{\left(-\frac{\partial A^\mu}{\partial t} - \frac{\partial \varphi}{\partial x^\mu}\right)}_{= E^\mu} + \frac{e}{c} v^\nu \underbrace{\left(\frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu}\right)}_{= \, ?}$$

(here, also summation over repeated indices will be understood.)

A bit of math needed here...

(perhaps, known?)

let $M_{\mu\nu}$ be a 3×3 matrix

s.t. $M_{\mu\nu} = -M_{\nu\mu}$ = antisymmetric

$$\| M_{\mu\nu} \| = \underbrace{\begin{pmatrix} 0 & M_{12} & M_{13} \\ -M_{12} & 0 & M_{23} \\ -M_{13} & -M_{23} & 0 \end{pmatrix}}$$

by antisymmetry, only 3-independent components (clearly so)

Introduce $\underline{\epsilon_{\mu\nu\lambda}}$ = "3-index totally antisymmetric tensor"

/ a useful bookkeeping device

$\epsilon_{\mu\nu\lambda} = 0$ if any of its indices are same

(e.g. $\epsilon_{112} = 0$, $\epsilon_{131} = 0$ ...)

$\epsilon_{123} = +1$ - by convention

$\epsilon_{213} = -\epsilon_{123} = -1$

$\epsilon_{231} = -\epsilon_{213} = +1$ etc...

Why useful? $(\vec{C} \times \vec{D})_\mu = \epsilon_{\mu\nu\lambda} C_\nu D_\lambda$

check $\left(\vec{C} \times \vec{D}\right) = \underbrace{\epsilon_{123}}_{+1} C_2 D_3 + \underbrace{\epsilon_{132}}_{= -\epsilon_{123} = -1} C_3 D_2$

(or x)

only (2 terms contribute)

$$\left(\bar{C} \times \bar{D}\right)_1 = C_2 D_3 - C_3 D_2$$

$$\left(\bar{C} \times \bar{D}\right)_x = C_y D_z - C_z D_y \text{ , similarly}$$

$$\left(\bar{C} \times \bar{D}\right)_y = C_z D_x - C_x D_z$$

$$\left(\bar{C} \times \bar{D}\right)_z = C_x D_y - C_y D_x .$$

So let $\vec{B} = \vec{\nabla} \times \vec{A}$ (, recall ?)

write as $B_\mu = \epsilon_{\mu\nu\lambda} \dfrac{\partial}{\partial x^\nu} A_\lambda$

$$B_1 = \partial_2 A_3 - \partial_3 A_2 \qquad B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$B_2 = \partial_3 A_1 - \partial_1 A_3 \qquad \text{etc.}$$

$$B_3 = \partial_1 A_2 - \partial_2 A_1$$

Now, having $M_{\mu\nu} = -M_{\nu\mu}$ $(\nu\mu = 1,2,3)$

means $M_{\mu\nu} = \epsilon_{\mu\nu\lambda} C_\lambda$ $\longrightarrow$

$$\|M_{\mu\nu}\| = \begin{pmatrix} 0 & M_{12} & M_{13} \\ -M_{12} & 0 & M_{23} \\ -M_{13} & -M_{13} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_{123} C_3 & \epsilon_{132} C_2 \\ & 0 & \epsilon_{231} C_1 \\ & & 0 \end{pmatrix}$$

(*)

$$= \begin{pmatrix} 0 & +C_3 & -C_2 \\ -C_3 & 0 & C_1 \\ C_2 & -C_1 & 0 \end{pmatrix}$$

(A 3x3 antisymmetric matrix is equivalent to a 3vector)

$$M_{\mu\nu} = -M_{\nu\mu} \iff C_\mu$$

the relation can be "inverted"

(a) from (*) $C_3 = M_{12} = \epsilon_{312} M_{12} = \frac{1}{2}(\epsilon_{312} M_{12} + \epsilon_{321} M_{2}$

$C_2 = -M_{13} = \epsilon_{213} M_{13} = \frac{1}{2}(\epsilon_{213} M_{13} + \epsilon_{231} M_{31}$

$C_1 = M_{23} = \epsilon_{123} M_{23} = \frac{1}{2}(\epsilon_{123} M_{23} + \epsilon_{132} M$

so $$C_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda} M_{\nu\lambda}$$ (we've proven it!)

(b) A more direct, equivalent proof

$M_{\mu\nu} = \epsilon_{\mu\nu\lambda} C_\lambda$, claim $C_\lambda = \frac{1}{2} \epsilon_{\lambda\alpha\beta} M_{\alpha\beta} \Rightarrow$ check

$$M_{\mu\nu} = \epsilon_{\mu\nu\lambda} \frac{1}{2} \epsilon_{\lambda\alpha\beta} M_{\alpha\beta} = \frac{1}{2}(\epsilon_{\mu\nu\lambda} \epsilon_{\lambda\alpha\beta}) M_{\alpha\beta} = \rightarrow$$

$$\epsilon_{\mu\nu\lambda}\,\epsilon_{\lambda\alpha\beta} = \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}$$

(Prove!! – really obvious)

$$\rightarrow \; = \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) M_{\alpha\beta} =$$

$$= \frac{1}{2}\left(M_{\mu\nu} - M_{\nu\mu}\right) = \frac{1}{2}\left(M_{\mu\nu} + M_{\mu\nu}\right) = M_{\mu\nu}$$

Now <u>let</u> $M_{\mu\nu} = \frac{\partial}{\partial x^\mu} A^\nu - \frac{\partial}{\partial x^\nu} A^\mu$

$$B_\lambda = \frac{1}{2}\epsilon_{\lambda\mu\nu} M_{\mu\nu} \quad \left(\text{as per usual def of } \vec{B}\right)$$

$$\left(B_1 = \epsilon_{123} M_{23} = \partial_2 A_3 - \partial_3 A_2 \; \cdots \right)$$

$$\& \quad M_{\mu\nu} = \epsilon_{\mu\nu\varkappa} B_\varkappa$$

<u>Check:</u> $M_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\varkappa}\,\epsilon_{\varkappa\alpha\beta} M_{\alpha\beta} =$

$$= \frac{1}{2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) M_{\alpha\beta} = \frac{1}{2}\left(M_{\mu\nu} - M_{\nu\mu}\right) = M_{\mu\nu}$$

So $\longrightarrow$ back to EOM, p. 68, bottom:

$$m\frac{d}{dt}\left(\gamma v^\mu\right) = e E^\mu + \frac{e}{c} v^\nu \left(\frac{\partial}{\partial x^\mu} A^\nu - \frac{\partial}{\partial x^\nu} A^\mu\right) =$$

$$= e E^\mu - \frac{e}{c} v^\nu \epsilon_{\mu\nu\varkappa} B^\varkappa = e E^\mu + \frac{e}{c}\epsilon^{\mu\nu\varkappa} v^\nu B^\varkappa$$

So finally we have

$$m \frac{d}{dt}(\gamma \vec{v}) = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}$$

or

$$\boxed{\frac{d\vec{p}}{dt} = e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)}$$

($\vec{p}$ is of course the 3-vector part of p^i, $\vec{p} = m\gamma\vec{v}$)

EOM of
a relativistic particle in external $\vec{E}$ & $\vec{B}$ field.

risking it being boring, but followed from:

- relativistic invariance of free particle action & interaction w/ external 4-vector field

Symmetries rule!

"only" expt. needed was
-- well, particles exist
-- 4-vector field exists
-- min # of derivatives