We discussed
$$A^i = (\phi, \vec{A})$$
&
$$A_i \to A_i + \frac{\partial \chi}{\partial x^i} \quad , \text{ since } A_i = (\phi, -\vec{A})$$

$$\left.\begin{aligned} \phi &\to \phi + \frac{\partial}{c\partial t}\chi \\ -\vec{A} &\to -\vec{A} + \vec{\nabla}\chi \end{aligned}\right\} \quad \begin{aligned} \phi &\to \phi + \frac{1}{c}\frac{\partial}{\partial t}\chi \\ \vec{A} &\to \vec{A} - \vec{\nabla}\chi \end{aligned}$$

since $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi$

& $\vec{B} = \vec{\nabla}\times\vec{A}$

we see that
$$\vec{E}\left(A_i + \frac{\partial}{\partial x^i}\chi\right) = \vec{E}(A_i) + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\chi - \frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\chi$$
$$= \vec{E}(A_i)$$
&
$$\vec{B}(\vec{A} - \vec{\nabla}\chi) = \vec{\nabla}\times\vec{A} - \vec{\nabla}\times(\vec{\nabla}\chi) = \vec{\nabla}\times\vec{A}$$
$$= \vec{B}(\vec{A})$$

* so EM fields do not change when gauge potentials A_i are "gauge transformed"

* $\{A^i\}$ offers a redundant description of EM fields
* However L (δ H) can only be written locally in terms of A^i
  (i.e. as expression local in coordinates)

* Weyl's principle of "gauge invariance" & "minimal coupling very important heuristically for QFT ...

So we have

$$L = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} + \frac{e}{c}\vec{A}\vec{v} - e\phi$$

&, as usual $\mathcal{E} = \vec{v}\cdot\frac{\partial L}{\partial\vec{v}} - L$ is the energy ($\frac{d\mathcal{E}}{dt} = -\frac{\partial L}{\partial t}$, as usual if L depends explicitly on time, $\mathcal{E}$ is not conserved)

The energy is

$$\mathcal{E} = \underbrace{\frac{mc^2}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}}_{\substack{\text{this is for a}\\ \text{free particle, we already obtained}}} + \frac{e}{c}\vec{A}\vec{v} - \left[\frac{e}{c}\vec{A}\vec{v} - e\phi\right]$$

$$\mathcal{E} = \underbrace{\frac{mc^2}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}}_{\substack{\text{"kinetic"}\\ \mathcal{E}_{kin}}} + \underbrace{e\phi}_{\substack{\text{"potential"}\\ \mathcal{E}_{pot}}} ;$$

Generally, $\mathcal{E}$ is not conserved. ((The rate of change of $\mathcal{E}_{kin}$ is the fourth equation of motion $\underbrace{\frac{d}{dt}(m\vec{u}) = e\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)}_{\text{last three components}}$,

while $\frac{d}{dt}\mathcal{E}_{kin} = e\vec{E}\cdot\vec{v}$ [DIY!]

is really related to $\frac{d}{dt}(mu^0)$, since $mu^0 = p^0 = \frac{\mathcal{E}_{kin}}{c}$))

Now we can "bite the bullet" & study EOM in 4-vector notation.

<u>This will be the quickest way to learn about the transformation properties of $\vec{E}$ & $\vec{B}$ under $SO(1,3)$.</u>

We start now from action in "4d" form:

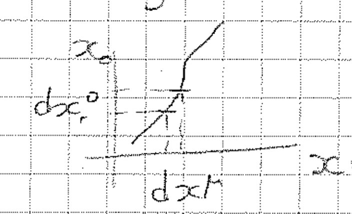
$$S = \int \left(-mc\, ds - \frac{e}{c} u^i A_i\, ds\right)$$

$$= \int \left(-mc\, ds - \frac{e}{c}\frac{dx^i}{ds} A_i\, ds\right)$$

$$= \int \left(-mc\, ds - \frac{e}{c}\, dx^i A_i\right)$$

where $\int$ is always along the worldline

$$S = \int_a^b \left(-mc\sqrt{dx^i dx_i} - \frac{e}{c} A_i\, dx^i\right)$$



$ds^2 = dx^i dx_i$

$\Rightarrow ds = \sqrt{dx^i dx_i}$

So now we want to vary the worldline: $x^i \to x^i + \delta x^i$, w/ $\delta x^i = 0$ @ beginning and end (at a & b).

$$\delta\sqrt{dx^i dx_i} = \frac{1}{\underbrace{2\sqrt{dx^i dx_i}}_{ds}}\, \delta\left(dx^k g_{kl}\, dx^l\right) =$$

$$= \underbrace{\frac{1}{2\,ds}\, 2\, dx^k}_{u^k} g_{kl}\, \delta dx^l = u_l\, \delta(dx^l) = u_l\, d(\delta x^l)$$

$$\delta A_i = \frac{\partial A_i}{\partial x^j}\, \delta x^j \text{, by chain rule}$$

So we have

$$\delta S = \int_a^b \Big(-mc\, \underbrace{u_\ell\, d\,\delta x^\ell}_{= d(u_\ell \delta x^\ell) - du_\ell \delta x_\ell} - \frac{e}{c} \overbrace{A_i\, d(\delta x^i)}^{= d(A_i \delta x^i) - (dA_i)\delta x^i} - \frac{e}{c} \frac{\partial A_i}{\partial x^\ell} \delta x^\ell dx^i \Big),$$

$$= \int_a^b \Big(-mc\, d(u_\ell \delta x^\ell) - \frac{e}{c} d(A_i \delta x^i)$$

$$+ mc\, du_\ell\, \delta x^\ell + \frac{e}{c} dA_i\, \delta x^i - \frac{e}{c} \frac{\partial A_i}{\partial x^\ell} dx^i \delta x^\ell \Big)$$

$$= \int_a^b d\Big[\Big(-mc\, u_\ell - \frac{e}{c} A_\ell\Big)\delta x^\ell\Big] \qquad dA_\ell = \frac{\partial A_\ell}{\partial x^i} dx^i$$

$$+ \int_a^b \delta x^\ell \Big[mc\, du_\ell + \frac{e}{c} dA_\ell - \frac{e}{c} \frac{\partial A_i}{\partial x^\ell} dx^i \Big] =$$

$$= \Big(-mc\, u_\ell - \frac{e}{c} A_\ell\Big)\delta x^\ell \Big|_{x=a}^{x=b}$$ vanishes since $\delta x^\ell = 0$ @ a & b

$$+ \int_a^b \delta x^\ell \Big[mc\, du_\ell + \frac{e}{c} \Big(\frac{\partial A_\ell}{\partial x^i} - \frac{\partial A_i}{\partial x^\ell} \Big) dx^i \Big]$$

$$= \int_a^b \delta x^\ell \Big(mc \frac{du_\ell}{ds} + \frac{e}{c} \Big(\frac{\partial A_\ell}{\partial x^i} - \frac{\partial A_i}{\partial x^\ell} \Big) \frac{dx^i}{ds} \Big) ds =$$

$= \delta S = 0 \Rightarrow$ by main theorem of variational calculus, since δx^ℓ is arbitrary, we must have

$$mc \frac{du_\ell}{ds} = -\frac{e}{c} \frac{dx^i}{ds} \Big(\frac{\partial A_\ell}{\partial x^i} - \frac{\partial A_i}{\partial x^\ell} \Big)$$

or $$\frac{d}{ds}\underbrace{\left(mc\, u_\ell\right)} = + \frac{e}{c}\underbrace{\left(\frac{\partial A_i}{\partial x^\ell} - \frac{\partial A_\ell}{\partial x^i}\right)} u^i$$

$$\frac{d}{ds}\, p_\ell = \frac{e}{c}\, F_{\ell i}\, u^i$$

as usual, $p_\ell = mc u_\ell$ $\quad (p^\ell = mc u^\ell)$

and we defined <u>4-tensor of 2nd rank</u> $\underline{F_{\ell i}}$

$$F_{\ell i} \equiv \frac{\partial A_i}{\partial x^\ell} - \frac{\partial A_\ell}{\partial x^i}$$

note $F_{\ell i} = - F_{i \ell}$, <u>it is an antisymmetric 4×4 matrix</u>

$A_i \equiv$ "<u>gauge potential</u>" (4-vector)
electromagnetic

$F_{\ell i} = \dfrac{\partial A_i}{\partial x^\ell} - \dfrac{\partial A_\ell}{\partial x^i}$ = "<u>field strength tensor</u>"
electromagnetic

$$F_{0\alpha} \equiv \frac{\partial A_\alpha}{\partial x^0} - \frac{\partial A_0}{\partial x^\alpha}, \quad A^i = (\phi, \vec{A}); \quad A_i = (\phi, -\vec{A})$$

this means $F_{01} = -\dfrac{\partial (\vec{A})_1}{\partial (ct)} - \dfrac{\partial \phi}{\partial x^1} = E_1$

$$F_{02} = -\frac{\partial (\vec{A})_2}{\partial (ct)} - \frac{\partial \phi}{\partial x^2} = E_2$$

$$F_{03} = -\frac{\partial (\vec{A})_3}{\partial (ct)} - \frac{\partial \phi}{\partial x^3} = E_3$$

as for $F_{\alpha\beta}$, we have

$$F_{\alpha\beta} = \frac{\partial}{\partial x^\alpha} A_\beta - \frac{\partial}{\partial x^\beta} A_\alpha$$

$$F_{12} = \frac{\partial}{\partial x^1}\left(-\vec{A}\right)_2 - \frac{\partial}{\partial x^2}\left(-(\vec{A})_1\right) = \frac{\partial(\vec{A})_1}{\partial x^2} - \frac{\partial(\vec{A})_2}{\partial x^1} = -B$$

recall $B_3 = \frac{\partial}{\partial x^1} A_2 - \frac{\partial}{\partial x^2} A_1$

$$F_{13} = \frac{\partial}{\partial x^1}\left(-(\vec{A})_3\right) - \frac{\partial}{\partial x^3}\left(-(\vec{A})_1\right) = \frac{\partial}{\partial x^3}(\vec{A})_1 - \frac{\partial}{\partial x^1}(\vec{A})_3 = B$$

$$B_2 = \frac{\partial}{\partial x^3} A_1 - \frac{\partial}{\partial x^1} A_3$$

$$F_{23} = \frac{\partial}{\partial x^2}\left(-(\vec{A})_3\right) - \frac{\partial}{\partial x^3}\left(-(\vec{A})_2\right) = \frac{\partial(\vec{A})_2}{\partial x^3} - \frac{\partial(\vec{A})_3}{\partial x^2} = -B$$

$$B_1 = \frac{\partial}{\partial x^2} A_3 - \frac{\partial}{\partial x^3} A_2$$

So ...

$$F_{ij} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

What is F^{ij}? : remember, every lifted spatial index gets a $-$ sign

So we have

$$F_{ij} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \longrightarrow F^{ij} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix},$$

also

$$\underbrace{F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} B_\gamma}$$

$$F_{12} = -\epsilon_{123} B_3 = -B_3$$

$$F_{13} = -\epsilon_{132} B_2 = +B_2$$

$$F_{23} = -\epsilon_{231} B_1 = -B_1$$

$$F^{0\alpha} = -F_{0\alpha}$$

$$F^{\alpha\beta} = +F_{\alpha\beta}$$

in, short

Using this map $F_{ij} \longrightarrow \vec{E}, \vec{B}$ we can see that

$$mc\frac{du_i}{ds} = \frac{e}{c} F_{ij} u^j$$

gives, using $u_i = (\gamma, -\gamma\frac{\vec{v}}{c})$

$u^i = (\gamma, \gamma\frac{\vec{v}}{c})$

when $i = 1, 2, 3$

$$-mc\frac{d}{ds}\left(\gamma\frac{v_\alpha}{c}\right) = \frac{e}{c}\left(F_{\alpha 0} u^0 + F_{\alpha\beta} u^\beta\right) =$$

$$= \frac{e}{c}\left(-E_\alpha \gamma - \epsilon_{\alpha\beta\mu} B_\mu \gamma \frac{v_\beta}{c}\right)$$

or

$$\frac{1}{c\sqrt{1-\frac{v^2}{c^2}}}\frac{d}{dt}(\gamma m v_\alpha) = \frac{e}{c}\left(E_\alpha + \epsilon_{\alpha\beta\mu}\frac{v_\beta}{c} B_\mu\right)\frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}$$

or finally,

$$\frac{d}{dt}(m\gamma\vec{v}) = e\left(\vec{E} + \vec{v}\times\vec{B}\,\frac{1}{c}\right) \quad (1)$$

the familiar E.o.M.; (p. 72-73)

the $i = 0$ component says:

$$mc\,\frac{du_0}{ds} = \frac{e}{c}\,F_{0\alpha}\,u^{\alpha}$$

$$\Downarrow$$

$$\frac{d(mc\gamma)}{c\sqrt{1-\frac{\vec{v}^2}{c^2}}\;dt} = \frac{e}{c}\,E_\alpha\,\gamma\,\frac{v^\alpha}{c} = \frac{e}{c^2}\,\vec{E}\cdot\vec{v}\,\frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}$$

$$\frac{d}{dt}(mc\gamma) = \frac{e}{c}\,\vec{E}\cdot\vec{v}$$

or

$$\frac{d}{dt}(mc^2\gamma) = e\,\vec{E}\cdot\vec{v}$$

$$\frac{d}{dt}(\mathcal{E}_{kin}) = e\,\vec{E}\cdot\vec{v} \quad (2)$$

$\underbrace{(1)\ \&\ (2)}$

one 4-vector equation

$$\frac{dp_i}{ds} = \frac{e}{c}F_{ij}v^j$$

MORAL

(change of kinetic energy of particle $= mc^2\gamma$) = (work done by Lorentz force)

$\vec{F}\cdot\vec{v}$; magnetic field does no work.

Now that we know

$$F^{ij} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & B_1 \\ E_3 & -B_2 & -B_1 & 0 \end{pmatrix} \equiv \hat{F}$$

under Lorentz transforms, F^{ij} transforms like the product of two 4-vectors, Best shown in matrix notation

$$\hat{X}' = \hat{O} \hat{X} \qquad \hat{X} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$\hat{O}$ → SO(1,3) matrix

$$\hat{F}' = \hat{O} \hat{F} \hat{O}^T \quad \text{where } \hat{F}$$

<u>HW</u> <u>derive transforms for $\vec{E}$ & $\vec{B}$ this way</u> (part of <u>HW 3</u>)

// side comment for mathematically educated people —

$A^i \Longleftrightarrow$ "one-form" $A = A_i \, dx^i$

$F^{ij} \Longleftrightarrow$ "two-form"

$$F = dA = d(A_i \, dx^i) =$$

$$= \frac{\partial A_i}{\partial x^j} dx^j \wedge dx^i$$

$$= \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) dx^j \wedge dx^i$$

$$= \frac{1}{2} F_{ji} \, dx^j \wedge dx^i$$

Gauge transform: $A = A' + d\chi$

$$dA = dA' + d\,d\,\chi$$

$$= dA'$$

$$\Rightarrow \quad F = F'$$

(using differentia forms very simple & very natural! way to study "gauge theories" & gravity //)

- E&M
- weak & strong interactions

all described by "gauge theories"