<u>Lorentz transforms of $\vec{E}$ & $\vec{B}$ are part of HW3</u>

(they follow from transforms of F^{ik})

Now it useful to consider what Lorentz-invariant combinations of $\vec{E}$ & $\vec{B}$ can be built.

logic: + $F_{ij} \ni (\vec{E}, \vec{B})$

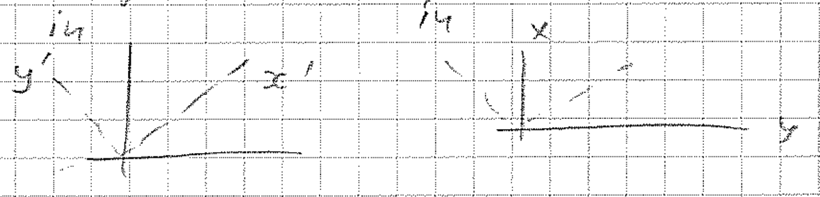
+ need to built the analogue of "dot products" involving F_{ij}, or F^{ij}, & g_{ij} ... that will be automatically same in any frame

Analogy: $\vec{E}\cdot\vec{E}$ is an invariant under rotations — so its value is same, no matter what frame orientation one considers

$$\vec{E}'\cdot\vec{E}' = \vec{E}\cdot\vec{E}$$

computed using components in

computed using components of $\vec{E}$ in



(For 3-vectors, we have, basically $\vec{A}\cdot\vec{B}$ & $(\vec{A}\times\vec{B})\cdot\vec{C}$
as the basic invariants $\vec{A}\cdot\vec{C}$
$\vec{B}\cdot\vec{C}$ + squares —)

From $\| F^{ij} \|$ there are two (and only two) Lorentz invariants:

$$\cdot \; F^{ij} F^{kl} g_{ik} g_{jl} \sim \vec{B}^2 - \vec{E}^2$$

&

$$\cdot \; F^{ij} F^{kl} \epsilon_{ijkl} \sim \vec{E} \cdot \vec{B}$$

(HW 3 again

$\sim$ means coeffs 1 or $1/2$ or so

NB: one could try

① $F^{ij} g_{ij}$, but this $\equiv 0$, since $F^{ij} = -F^{ji}$ while $g_{ij} = +g_{ji}$ (HW 3 again)

② ϵ_{ijkl} is the 4-dim' generalization of $\epsilon_{\alpha\beta\gamma}$. Conventions:

$$\epsilon^{ijkl} = 0, \text{ if any two indices equal}$$

$$\epsilon^{0123} = +1$$

$$\epsilon_{0123} = -1, \text{ since } \epsilon_{0123} = g_{00} \, g_{11} \, g_{22} \, g_{33} \, \epsilon^{0123}$$

$$(1 \times -1 \times -1 \times -1 \times +1)$$

(HW 3: exercises w/ ϵ_{ijkl}.)

While $\vec{E}$ & $\vec{B}$ (& $\vec{E}^2$ & $\vec{B}^2$)
are <u>NOT</u> Lorentz inv't,
$E^2 - B^2$ & $\vec{E} \cdot \vec{B}$ are.

important implications:

- if $\vec{E} \perp \vec{B}$ in one frame ($\vec{E} \cdot \vec{B} = 0$)
  then $\vec{E} \perp \vec{B}$ in all frames
- if $|\vec{E}| = |\vec{B}|$ in one frame, then
  $|\vec{E}| = |\vec{B}|$ in all frames
- furthermore, if $E^2 - B^2 > 0$ & $\vec{E} \perp \vec{B}$
  in one frame, then there exists a
  frame where $\vec{B} = 0$ (useful to simplify
  solving problems of motion in such $\vec{E}$ & $\vec{B}$)

(recall problem (4) in HW 2)

- or, we can always find a frame where
  $\vec{E}$ & $\vec{B}$ are parallel . . .

Knowing the invariants will also be very useful
when trying to write the relativistic invariant
action for the 4-vector field --- our next topic!

We learned that $F_{ij} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i$

& that $F_{0\alpha} = E_\alpha$

$$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} B_\gamma \, .$$

This fact already contains "one-half" of the familiar Maxwell equations ...

since
$$F_{ij} = \partial_i A_j - \partial_j A_i$$

$$\underbrace{\partial_i A_j}_{\text{a convenient shortcut}} \equiv \frac{\partial}{\partial x^i} A_j \quad \text{etc}$$

note that this implies

$$\epsilon^{iklm} \partial_k F_{lm} = \epsilon^{iklm} \partial_k (\partial_l A_m - \partial_m A_l)$$

$$= \epsilon^{iklm} \partial_k \partial_l A_m - (\epsilon^{ik'l'm'}) \partial_{k'} \partial_{m'} A_{l'} =$$

I switched indices, so got a $\times(-1)$

$$= \epsilon^{iklm} \partial_k \partial_l A_m - (-\epsilon^{ik'm'l'}) \partial_{k'} \partial_{m'} A_{l'}$$

$$= \epsilon^{iklm} \partial_k \partial_l A_m + \epsilon^{ik'm'l'} \partial_{k'} \partial_{m'} \partial_{l'}$$

rename "dummy" indices: $m' \to m$, $l' \to l$, $k' \to k$

$$= 2\epsilon^{iklm} \partial_k \partial_l A_m =$$

$$= \epsilon^{iklm} \partial_k \partial_l A_m \underbrace{- \epsilon^{ilkm}}_{(= + \epsilon^{iklm} \quad \dots)} \partial_k \partial_l A_m = \longrightarrow$$

$$\longrightarrow\ = \epsilon^{iklm}\,\partial_k \partial_l A_m - \epsilon^{ilkm}\,\underbrace{\partial_l \partial_k}\, A_m$$

since derivatives commute

$$\partial_k \partial_l = \partial_l \partial_k$$

then rename dummy $lkm \to klm$

$$\begin{matrix} l & k & m \\ \downarrow & \downarrow & \downarrow \\ k & l & m \end{matrix}$$

$$= \epsilon^{iklm}\,\partial_k \partial_l A_m - \epsilon^{iklm}\,\partial_k \partial_l A_m =$$

$$= \epsilon^{iklm}\left(\partial_k \partial_l A_m - \partial_k \partial_l A_m\right) = 0 \quad !$$

(NB: this is the 4dim version of $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$.)

Upshot: $\epsilon^{iklm}\,\partial_k F_{lm} = 0$ ("Bianchi identity")

$\downarrow$ implies what? $\quad (d\,dA = 0)$

4 equations, really

$$0 = \epsilon^{0ijk}\,\partial_i F_{jk} = \partial_i\, \underbrace{\epsilon^{\overbrace{ijk}^{1,2,3 \text{ only}}} F_{jk}}$$

rename α, β, γ since 1,2,3 only

$$0 = \partial_\alpha\, \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} = \partial_\alpha\, \epsilon^{\alpha\beta\gamma}\left(-\epsilon_{\beta\gamma\delta}\, B_\delta\right) = -2\,\partial_\alpha B_\alpha =$$

$$= -2\,\frac{\partial}{\partial x^\alpha} B_\alpha = 2\vec{\nabla}\cdot\vec{B} = 0$$

$$\epsilon^{\alpha\beta\gamma}\epsilon_{\beta\gamma\delta} = 2\,\delta^\alpha{}_\delta \quad (\text{HW3})$$

$$\epsilon^{0ijk}\partial_i F_{jk} = 0 \iff \vec{\nabla}\cdot\vec{B} = 0$$

(≡ no magnetic charges)

<u>What about the other 3 relations?</u>

$$0 = \epsilon^{\alpha ijk}\partial_i F_{jk} = \epsilon^{\alpha 0\beta\gamma}\partial_0 F_{\beta\gamma} \to -\epsilon^{\alpha\beta\gamma}\partial_0 F_{\beta\gamma}$$

$$+\epsilon^{\alpha\beta 0\gamma}\partial_\beta F_{0\gamma} \to +\epsilon^{\alpha\beta\gamma}\partial_\beta F_{0\gamma}$$

$$+\epsilon^{\alpha\beta\gamma 0}\partial_\beta F_{\gamma 0} \to -\epsilon^{\alpha\beta\gamma}\partial_\beta F_{\gamma 0} =$$

$$= +\epsilon^{\alpha\beta\gamma}\partial_\beta F_{0\gamma}$$

$$(F_{\gamma 0} = -F_{0\gamma})$$

$$= 2\,\epsilon^{\alpha\beta\gamma}\partial_\beta F_{0\gamma} - \epsilon^{\alpha\beta\gamma}\partial_0 F_{\beta\gamma}$$

$$= 2\,\epsilon^{\alpha\beta\gamma}\partial_\beta E_\gamma - \epsilon^{\alpha\beta\gamma}\partial_0(-\epsilon_{\beta\gamma\delta}B_\delta)$$

$$= 2\,\epsilon^{\alpha\beta\gamma}\partial_\beta E_\gamma + \underbrace{\epsilon^{\alpha\beta\gamma}\epsilon_{\beta\gamma\delta}}\,\partial_0 B_\delta$$

$$0 = 2\,\epsilon^{\alpha\beta\gamma}\partial_\beta E_\gamma + 2\delta^\alpha_\delta\,\partial_0 B_\delta$$

so: $\quad 0 = \epsilon^{\alpha\beta\gamma}\partial_\beta E_\gamma + \partial_0 B_\alpha$

$$\frac{1}{c}\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times\vec{E} = 0$$

hence Bianchi identity $\quad \epsilon^{ijkl}\partial_j F_{kl} = 0$

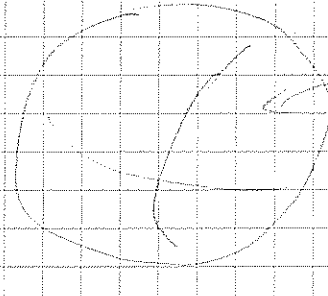
$$\Updownarrow$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \& \quad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

(no magnetic charge) $\qquad$ (changing $\vec{B}$ → creates $\vec{E}$)

In integral form

$$\vec{\nabla}\cdot\vec{B} = 0 \iff$$

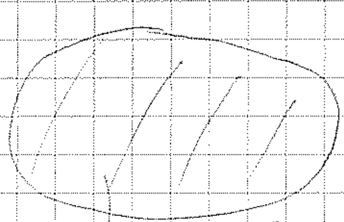


$S = \partial V$

closed surface binding a Volume

$$\int_V \vec{\nabla}\cdot\vec{B}\, d^3x = 0 \xrightarrow{\text{integrate by parts}} \oint d^2\vec{S}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \iff$$



$$\int_S \vec{\nabla}\times\vec{E}\cdot d^2\vec{S} = \oint_{\partial S = L} \vec{E}\cdot d\vec{l}$$

$\partial S = L$

S (Surface)

(L - line binding surface)

so

$$\oint_{\partial S = L} \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int_S \vec{B}\cdot d^2\vec{S}$$

circulation of $\vec{E}$ on $L = \partial S$ $= -\frac{1}{c}\frac{d}{dt}$ (flux of $\vec{B}$ thru S ($\partial S = L$))

Hence two of the 4 Maxwell equations are simply a consequence of $F_{ij} = \partial_i A_j - \partial_j A_i \Rightarrow$ the Bianchi identity $\epsilon^{ijkl}\partial_j F_{kl} = 0$

(so there's no dynamics involved, but pure "kinematics" — or gauge invariance's simple consequence)

? about other two?

To get the rest of Maxwell eqns, we need more ---

relativity, of course, so E.O.M. will be same in any frame

more is needed --- in classical physics we believe we have particles & "waves"

or, rather, "fields"

there, we already know how to describe in an SO(1,3) invariant manner, including their interaction w/ fields

$$S = S_{matter} + S_{inter.}$$

"external field"

particles (that is)

$$\left(-mc \int ds \right)$$

$$\left(-\frac{e}{c} \int ds\, u^i A_i \right)$$

but fields are their own entity; they propagate on their own once created

so they are also dynamical degrees of freedom — albeit a bit more unusual. there's a degree of freedom at every point in space: $A^i(\vec{x}, t)$

point in space

How can we write a Lagrangian (or action) for this continuum of degrees of freedom

(since $\forall\, \vec{x} \to$ a 4-vector $A^i(\vec{x}, t)$)

- Since the field does not have a position (or a worldline) associated with it, the action has to be an integral over the whole of space time: $S_{\text{field}} = \int d^3x\, dt\, \underbrace{\mathcal{L}(\vec{x}, t)}$ of a local function of the field degrees of freedom $(A^i(\vec{x},t))$ → the Lagrangian of the degrees of freedom at the given $\vec{x}$

- Relativity will impose $\mathcal{L}(\vec{x}, t)$ be a Lorentz scalar
- Gauge-invariance will require that $\mathcal{L}(\vec{x}, t)$ depend only on F_{ij}, but not on A_i (if it depended on A_i in a non-gauge-invariant manner, then this would describe more than $\vec{E}$ & $\vec{B}$ —)

- Superposition principle requires that EOM for the field itself be linear, so that a Σ of solutions is a solution. This is an experimental fact for E&M fields.

In summary,

$$S_{field} \sim \int d^3x\, dt \Big(\text{Lorentz invariant function of } F_{ij} \text{ (at } \bar{x}, t) \text{ which will give linear EOM}\Big).$$

For a field $A^i(\bar{x}, t)$, the dynamical variables are the values of the field $\forall \bar{x}, \forall t$. Variational principle would require that S_{field} be stationary w.r.t. arbitrary variation $\delta A^i(\bar{x}, t)$ of $A^i(x, t)$. Hence, linear EOM will only result if $\mathcal{L}$ is quadratic wrt A^i (& hence F^{ij}) (just as $\frac{m\vec{v}^2}{2} - \frac{\omega^2}{2} m \vec{x}^2$ leads to linear equations)

So, we're left w/ no choice:

$$(1) \quad S_{field} \sim \int d^3x\, dt\, F^{ij} F^{kl} g_{ik} g_{jl}$$

and/or

$$(2) \quad S_{field} \sim \underbrace{\int d^3x\, dt\, F^{ij} F^{kl} \epsilon_{ijkl}}_{\text{this is a "boundary term" (HW.3)}}$$

not good: which can not affect EOM

Thus we have no choice but accept

$$\mathcal{L} \sim F^{ij} F_{ij} .$$

Let $d^4x = dx^0\, dx^1\, dx^2\, dx^3 = c\,dt\, d^3\vec{x}$

$$S_{\substack{E\&M \\ field}} \stackrel{!}{=} -\frac{1}{16\pi c} \int d^4x\, F^{ij} F_{ij}$$

so that $\left(\frac{\partial \vec{A}}{\partial t}\right)^2$ appears w/ "+" sign in action

$$= \frac{(+1)\, 1}{8\pi} \int d^3\vec{x}\, dt \left(\vec{E}^2 - \vec{B}^2 \right) \qquad \left(F^{ij}F_{ij} = \frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right) \right)$$

this is a choice of units. Gaussian system.

$\varphi = \frac{q}{r}$ ← Coulomb field of charge q

In full glory we have

$$\boxed{\begin{aligned} S &= S_{matter} + S_{inter} + S_{EM\ field} \\ S &= -mc\int ds - \frac{e}{c}\int ds\, u^i A_i - \frac{1}{16\pi c}\int d^4x\, F^{ij}F_{ij} \end{aligned}}$$

Great! What are its consequences?

Complicated mess: charges (particles) are sources of E&M field and are influenced by the E&M field, incl. their own field ...

-- we'll see that classical relativistic E&M does not come fully to grips w/ "solving" this system (not in all circumstances), and, as usual for good theories in physics, is able to "foresee" its "death", i.e. limitations. --

<u>Our next tasks are many</u> -

- what are the EOM? -- Maxwell's, as we'll see
- what are the conserved quantities?
  - ↳ EM field carries momentum, energy, & angular momentum
- EM waves & radiation - and interaction of particles w/ them. -

The dynamical variables that appear in the action

$$S = -mc\int ds - \frac{e}{c}\int ds\, u^i A_i - \frac{1}{16\pi c}\int d^4x\, F^{ij}F_{ij}$$

are (1) the particles' position (worldlines - $x^i_A(\tau)$)

meaning there could be more than one particle

i.e.

$$S_{\substack{\text{matter +}\\ \text{+ interaction}}} = \sum_A \Big(-\underbrace{m_A}_{\text{A-th mass}} c \underbrace{\int ds}_{\text{A-th worldline}} - \underbrace{\frac{e_A}{c}}_{\text{A-th charge}} \underbrace{\int ds}_{\text{A-th worldline}} \underbrace{u^i_A}_{\text{A-th velocity}} A_i \Big)$$

$\sum_A$: over all particles w/ masses m_A & charges q_A

The particles' positions $x_A^i(\tau)$ only appear in $S_{matter} + S_{interin}$. Hence varying the action w.r.t. $\delta x_A^i(\tau)$ we obtain the equation of motion for each particle:

$$\begin{cases} \dfrac{d\vec{p}_A}{dt} = e\left(\vec{E}(\vec{x}_A, t) + \dfrac{\vec{v}_A}{c} \times \vec{B}(\vec{x}_A, t)\right) \\ \text{for all } A \end{cases}$$

On the other hand, the 4-vector potential A_i only appears in $S_{interaction}$ as well as in $-\frac{1}{16\pi c}\int d^4x\, F^{ij}F_{ij}$. Moreover, A_i "couples" in a rather uniform way to all particles:

$$S_{inter} = \sum_A \Big(-\frac{e_A}{c}\underset{\substack{\uparrow \\ (x_A^i(\tau))}}{\int} ds\, u_A^i \underset{\substack{\uparrow \\ \text{value of } A_i \text{ at 4-position} \\ \text{of } A\text{-th particle}}}{A_i(\{x_A^2\})}\Big)$$

It is very advantageous (convenient, also leads to something you are familiar with) to rewrite this term as follows:

$$S_{int} = -\frac{1}{c^2}\int d^4x\, A^i(x)\, j_i(x),$$

where

$$j^i(x) = \sum_A c e_A \underset{[A\text{-th worldline } x_A^i(\tau)]}{\int} ds\, u_A^i\, \delta(x^0 - x_A^0(\tau))\,\delta(x^1 - x_A^1(\tau))\,\delta(x^2 - x_A^2(\tau)) \times \delta(x^3 - x_A^3(\tau)).$$

"4-vector of electromagnetic current"

The next few pages will go into elucidating this notation = "4-current"

Imagine there's only one particle

$$j^i(x) = ec\int ds\, u^i \overbrace{\delta(x^0 - x^0(\tau))\,\delta(x^1 - x^1(\tau))\,\delta(x^2 - x^2(\tau))\,\delta(x^3 - x^3(\tau))}^{\equiv \delta^4(x - x(\tau))}$$

(worldline $x^i(\tau)$) ; $ds\, u^i = ds \frac{dx^i(\tau)}{ds} = dx^i(\tau)$

then $$-\frac{1}{c^2}\int d^4x\, A_i(x)\, j^i(x) = -\frac{e}{c}\int d^4x\, A_i(x) \int dx^i(\tau)\, \delta^{(4)}(x - x(\tau)) =$$

$$= -\frac{e}{c}\int dx^i(\tau) \int d^4x\, A_i(x)\, \delta^{(4)}(x - x(\tau))$$

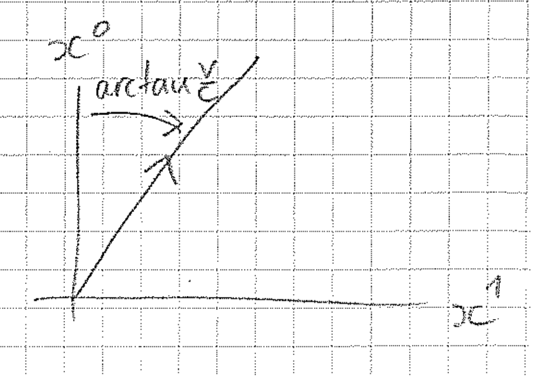
$$= -\frac{e}{c}\int dx^i(\tau) \underbrace{\int dx^0 dx^1 dx^2 dx^3\, A_i(x^0, x^1, x^2, x^3)\, \delta(x^0 - x^0(\tau))\, \delta(x^1 - x^1(\tau)) \times \delta(x^2 - x^2(\tau))\, \delta(x^3 - x^3(\tau))}_{\text{effect of 4 } \delta \text{ functions: replace } x^i \to x^i(\tau),\ i = 0,1,2,3}$$

$$= -\frac{e}{c}\int dx^i(\tau)\, A_i(x(\tau)) \quad \text{— usual } S_{int}$$

More concisely written, 4-current for one particle is

$$j^i(x) = ce \int_{\substack{\text{worldline} \\ \{x^i(\tau)\}}} dx^i(\tau)\, \delta^4(x - x(\tau)).$$

Consider following simple worldline



i.e a particle moving w/ uniform velocity.

the worldline $x^i(\tau)$ is:

$$\left.\begin{aligned} x^0(\tau) &= c\tau \\ x^1(\tau) &= v\tau \\ x^2(\tau) &= 0 \\ x^3(\tau) &= 0 \end{aligned}\right\}$$ I assumed particle is at $\vec{x}=0$ at $x^0=0$..

$\tau \in (-\infty, \infty)$

then $$\left.\begin{aligned} dx^0(\tau) &= c\, d\tau \\ dx^1(\tau) &= v\, d\tau \\ dx^2(\tau) &= dx^3(\tau) = 0 \end{aligned}\right\}$$ along worldline

Hence

$$j^i(x) = ec\int dx^i(\tau)\, \delta^{(4)}(x - x(\tau))$$ is

$j^2 = j^3 = 0$ since $dx^2 = dx^3 = 0$ along chosen worldline

$$j^0(x) = ec^2 \int_{-\infty}^{\infty} d\tau\, \delta(x^0 - c\tau)\, \delta(x^1 - v\tau)\, \delta(x^2)\, \delta(x^3)$$

&

$$j^1(x) = ecv \int_{-\infty}^{\infty} d\tau\, \delta(x^0 - c\tau)\, \delta(x^1 - v\tau)\, \delta(x^3)\, \delta(x^2)$$

hence

$$\begin{pmatrix} j^0(x) \\ j^1(x) \\ j^2(x) \\ j^3(x) \end{pmatrix} = e \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} c\, \delta(x^2)\, \delta(x^3) \underbrace{\int_{-\infty}^{\infty} d\tau\, \delta(x^0 - c\tau)\, \delta(x^1 - v\tau)}_{?}$$

BEGIN MATH INTERLUDE

We know: $\int_{-\infty}^{\infty} dx\, \delta(x)\, g(x) = g(0)$; & $\underbrace{\delta(-x) = \delta(x)}_{\text{it's EVEN!}}$

$\Rightarrow$ the DEF. of $\delta(x)$

Similarly, $\int_{-\infty}^{\infty} dx\ \delta(x-a)\, g(x) = \left[\begin{array}{c} x = y + a \\ dx = dy \end{array} \right]$

$$= \int_{-\infty}^{\infty} dy\ \delta(y)\, g(y+a)$$

$$= g(a).$$

What about $\int_{-\infty}^{\infty} dx\ \delta(c(x-a))\, g(x) = \circledast$?

let $x = \frac{y}{|c|} + a$

$dx = \frac{1}{|c|} dy$

$$\circledast = \frac{1}{|c|} \int_{-\infty}^{\infty} dy\ \delta(y)\, g\left(\frac{y}{|c|} + a\right) \Rightarrow$$

$$\int_{-\infty}^{\infty} dx\ \delta(c(x-a))\, g(x) = \frac{1}{|c|}\, g(a)$$

Finally, $\int_{-\infty}^{\infty} dx\ \underbrace{\delta(f(x))}\, g(x) = ?$

clearly, only x_* where $f(x_*) = 0$ matters

but $f(x - x_*) \approx f'(x_*)(x - x_*)$ near $x = x_*$

so $\int_{-\infty}^{\infty} dx\ \delta(f(x))\, g(x) = \left(\begin{array}{l} \text{assume unique } x_* \text{ where } f(x_*) = 0 \\ \text{\& } f'(x_*) \neq 0 \end{array} \right)$

$$= \int_{-\infty}^{\infty} dx\ \delta(\underbrace{f'(x_*)}_{c}(x - \underbrace{x_*}_{a}))\, g(x) = \frac{1}{|f'(x_*)|}\, g(x_*)$$

Mural: $$\int_{-\infty}^{\infty} dx \; \delta(f(x)) \, g(x) =$$

$$= \sum_{x^*:\; f(x_*)=0} \frac{g(x_*)}{|f'(x_*)|}$$

or, formally:

$$\left\{ \begin{aligned} \delta(cx) &= \frac{1}{|c|}\,\delta(x) \\ \delta(f(x)) &= \sum_{\substack{x^*: \\ f(x_*)=0}} \frac{\delta(x-x_*)}{|f'(x_*)|} \end{aligned} \right\}$$

⇑ END OF THIS MATH INTERLUDE ⇑

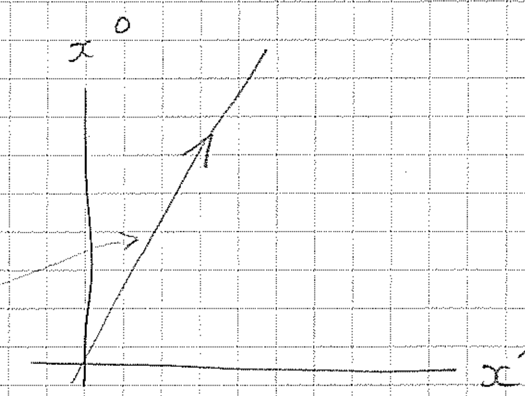
We left @ p-98, having to do

$$\int_{-\infty}^{\infty} d\tau \; \underbrace{\delta(x^0 - c\tau)}_{\left(=\frac{1}{c}\delta\left(\tau - \frac{x^0}{c}\right)\right)} \; \underbrace{\delta(x^1 - v\tau)}_{\hookrightarrow \text{ treat as } g(\tau)} =$$

$$= \int_{-\infty}^{\infty} d\tau \, \frac{1}{c}\delta\left(\tau - \frac{x^0}{c}\right) \delta(x^1 - v\tau) = \frac{1}{c}\,\delta\left(x^1 - v\frac{x^0}{c}\right)$$

hence $$\begin{pmatrix} j^0 \\ j^1 \\ j^2 \\ j^3 \end{pmatrix}(x) = e \begin{pmatrix} c \\ v \\ 0 \\ 0 \end{pmatrix} \delta(x^2)\,\delta(x^3)\,\delta\left(x^1 - v\frac{x^0}{c}\right)$$

$$\begin{cases} j^0 = c\ e\ \delta(x^2)\,\delta(x^3)\,\delta\!\left(x^1 - v\,\frac{x^0}{c}\right) \\ j^1 = v\ e\ \delta(x^2)\,\delta(x^3)\,\underbrace{\delta\!\left(x^1 - v\,\frac{x^0}{c}\right)} \\ j^2 = j^3 = 0 \end{cases}$$



$x^2 = x^3 = 0$ slice

δ fn $\neq 0$ only on worldline

Now, $e\ \delta(x^2)\,\delta(x^3)\,\delta(x^1 - \frac{v}{c}x^0) =$

$$= e\ \underbrace{\delta(x^2)\,\delta(x^3)\,\delta(x^1 - vt) = \rho(x^1, x^2, x^3, t)}$$

charge density $\neq 0$ only on worldline $\left(\int d^3x\ \rho(x^1 x^2 x^3 t) = e \quad \text{total charge}\right)$

hence $\quad j^0 = c\rho$

& $\quad j^1 = v\rho \longrightarrow$ velocity $\times$ charge density =

= charge current =

(= amount of charge going thru unit area $\perp$ x^1 in unit time

Hence, physical meaning of 4-current

$$j^i(x) = \sum_A c\, e_A \int dx^i_A(\tau)\ \delta^4(x - x_A(\tau)) \quad \text{is:} \quad j^i = (c\rho; \vec{j})$$

$j^0(x) = c \times$ (charge density $\rho(x)$); $\vec{j}(x) = \vec{v}(x) \times \rho(x) =$ (velocity $\times$ $\times$ charge density)

Writing $S_{int} = -\frac{1}{c^2}\int d^4x\, j^i(x)\, A_i(x)$

can be now interpreted as saying that particles couple to A_0 via their charge density and to $\vec{A}$ via their current density

-- taken another way: • charge density is a source of A_0

• charge current is a source of $\vec{A}$

We wrote $j^i(x)$ for a discrete distribution of charges moving on worldlines $\{x^i_A(\tau)\}$

$$j^0(x) = \sum_A e_A c \int dx^0_A(\tau)\, \delta^{(4)}(x - x_A(\tau))$$

$$\vec{j}(x) = \sum_A e_A c \int d\vec{x}_A(\tau)\, \delta^{(4)}(x - x_A(\tau))$$

When considering problem of creation of EM fields, it is often convenient to work w/ continuous charge & current densities, i.e. use $j^i = (c\rho, \vec{v}\rho)$, w/ ρ -- some local continuous distribution of charge

(i.e. in macroscopic media, magnetohydrodynamics (plasmas, etc --).