$$S = -\sum_a m_a c \int_a ds - \frac{q_a}{c} \int_a dx^i A_i(x)$$

$$- \frac{1}{16\pi c} \int d^4x \, F^{ij} F_{ij}$$

how can we deal w/ the combined

motion → radiation/e.m.-field → motion → $\frac{\text{rad'n}}{\text{e.m}}$ → ... ?

one possibility → can we solve for $A^i(x)$ for arbitrary positions of particles, & substitute this $A^i(x)$ back in the $S$ & study E.O.M. for particles?

<u>Formally</u> → yes! use linearity of Maxwell:

rewrite $S = \sum_a -m_a c \int_a ds - \frac{1}{c^2} \int d^4x \, j^i(x) A_i(x)$

$$\left( \begin{aligned} j^0 &= c \sum_a q^a \delta^{(3)}(\vec{x} - \vec{x}^a) \\ \vec{j} &= \sum_a q_a \vec{v}^a \delta^{(3)}(\vec{x} - \vec{x}^a) \end{aligned} \right) \quad - \frac{1}{16\pi c} \int d^4x \, F^{ij} F_{ij}$$

EOM for $A_i$:

$$\partial_i F^{ij} = \frac{4\pi}{c} j^j ,$$

choosing, say Lorentz gauge, we have

$$A^j(x) = \frac{1}{c} \int d^3x' dt' \, \frac{\delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|} \, j^j(\vec{x}', t')$$

let's try to see what this would be ... ②

use $j^{j}(x,t) = \sum_a q_a \begin{pmatrix} c \\ \vec{v}^{a}(t) \end{pmatrix} \delta^{(3)}(\vec{x} - \vec{x}^{a}(t))$

to find

$$A^{j}(\vec{x},t) = \frac{1}{c} \sum_a q_a \int d^3x' dt' \begin{pmatrix} c \\ \vec{v}^{a}(t') \end{pmatrix} \frac{\delta\left(t - t' - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|} \delta^{(3)}(\vec{x}' - \vec{x}^{a}(t'))$$

$$= \frac{1}{c} \sum_a q_a \int dt' \frac{\delta\left(t - t' - \frac{|\vec{x} - \vec{x}_a(t')|}{c}\right)}{|\vec{x} - \vec{x}_a(t')|} \begin{pmatrix} c \\ \vec{v}_a(t') \end{pmatrix}$$

$$A^{j}(\vec{x},t) = \sum_a q_a \begin{pmatrix} c \\ \vec{v}_a(t_r) \end{pmatrix} \frac{1}{c|\vec{x} - \vec{x}_a(t_r)| - \vec{v}_a(t_r) \cdot (\vec{x} - \vec{x}_a(t_r))}$$

$t_r$: depends on $\underline{x, t, \& \ a}$

we could anyway proceed and evaluate

$$-\frac{1}{c^2} \int d^4x \ j^{i} A_i - \frac{1}{16\pi c} \int d^4x \ F_{ij} F^{ij}$$

on this solution of $A_i$ EOM:

$$-\frac{1}{c} \int d^4x \ j^{i} A_i - \frac{1}{16\pi c} \int d^4x \ 2 \partial_i A_j F^{ij} = \text{(drop } \oint \text{ term)}$$

$$= -\frac{1}{c} \int d^4x \ j^{i} A_i + \frac{1}{8\pi c} \int d^4x \ A_j \underbrace{\partial_i F^{ij}}_{\frac{4\pi}{c} j^{j}} = -\frac{1}{2c} \int d^4x \ j^{i} A_i$$

thus, the combined particle + field action (on the solution of EOM for $A_i$) is ③

$$S = \sum_a - m_a c^2 \int dt \sqrt{1 - \frac{v_a^2}{c^2}}$$

$$- \frac{1}{2c^2} \int d^4x \; j^i A_i = \qquad d^4x = d^3x \, c\,dt$$

$$= \sum_a - m_a c^2 \int dt \sqrt{1 - \frac{v_a^2}{c^2}}$$

$$- \frac{1}{2c} \int d^3x \, dt \left( j^0 A^0 - \vec{j}\,\vec{A} \right) =$$

$$= \sum_a - m_a c^2 \int dt \sqrt{1 - \frac{v_a^2}{c^2}} - \frac{1}{2} \int d^3x \sum_a q_a \delta^{(3)}(\bar{x} - \bar{x}_a(t)) A_0(\bar{x}, t)$$

$$+ \frac{1}{2} \int d^3x \sum_a q_a \frac{\vec{v}_a(t)}{c} \delta^{(3)}(\bar{x} - \bar{x}_a(t)) \vec{A}(\bar{x}, t)$$

$$= \sum_a \Big[ - m_a c^2 \int dt \sqrt{1 - \frac{v_a^2}{c^2}} - \frac{1}{2} q_a A_0(\bar{x}_a(t), t)$$

$$+ \frac{1}{2} q_a \frac{\vec{v}_a(t)}{c} \cdot \vec{A}(\bar{x}_a(t), t) \Big]$$

hum ... but now recall $A_0(\bar{x}, t) = \sum_b \dfrac{q_b c}{c|x - x_b(t_r)| - \bar{V}_b(t_r) \cdot (\bar{x} - \bar{x}_b(t_r))}$

...

so we have

$$\mathcal{L} = \sum_a - m_a c^2 \sqrt{1 - \frac{v_a^2(t)}{c^2}}$$

$$- \frac{1}{2} \sum_a \sum_b \frac{q_a q_b}{|\vec{x}_a(t) - \vec{x}_b(t_r)| - \frac{\vec{v}_b(t_r)}{c} \cdot (\vec{x}_a(t) - \vec{x}_b(t_r))}$$

$$+ \frac{1}{2} \sum_a \sum_b \frac{q_a q_b \, \vec{v}_a(t) \cdot \vec{v}_b(t_r) \, \frac{1}{c^2}}{|\vec{x}_a(t) - \vec{x}_b(t_r)| - \frac{1}{c} \vec{v}_b(t_r) \cdot (\vec{x}_a(t) - \vec{x}_b(t_r))}$$

– Now this is one hell of a mess...

(1) $\mathcal{L}$ depends on values of $\vec{x}_{a,b}$'s @ different times!

each $t_r$ is different!

$t_r = t_r(\vec{x}_a(t), \vec{x}_b \text{ worldline})$

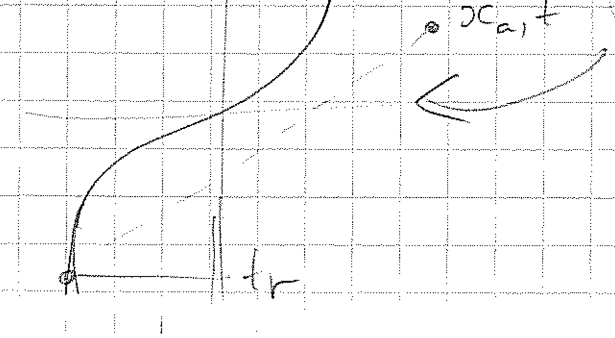
it is not 'local' in time, i.e. has q's @ diff't times..

(2) for $a = b$ terms there's even worse!

· $\infty$-ies all over the place...

$\vec{x}_a(t) - \vec{x}_b(t_r) = ?$

$x_b$ · $\bar{x}_{a,t}$ $t_r = t$ !!



<u>last time</u>: take $c \to \infty$

- all $t_r = t$
- all $|\bar{v}|/c$ term $\to 0$

except

$$-m_a c^2 \sqrt{1 - \frac{v_a^2}{c^2}} \to \frac{m_a v_a^2}{2}$$

$$\mathcal{L} = \sum_a - m_a c^2 + \sum_a \frac{m_a \bar{v}_a^2}{2} - \frac{1}{2} \sum_{a,b} \frac{q_a \, q_b}{|\bar{x}_a(t) - \bar{x}_b(t)|}$$

irrelevant constant ($\infty$)

this is OK --

if $a \neq b$ <u>kept</u> <u>only</u>

$$\mathcal{L} = \sum_a \frac{m_a v_a^2}{2} - \sum_{a>b} \frac{q_a \, q_b}{|\bar{x}_a - \bar{x}_b|}$$

Now this is one good $\mathcal{L}$

(most of 'everyday' stuff)

Can we do better?

**YES.** (a) To $\underline{O}\left(\frac{v^2}{c^2}\right)$ one can find a Local $\mathcal{L}$ (depending on $q$'s @ same time, so well formulated 'mechanics' problem)

(b) To $\underline{O}\left(\frac{v^3}{c^3}\right)$

there are issues – having to do w/ radiation. Since radiation is $\sim \frac{1}{c^3}$ effect, it means that EM field D.O.F. have to be included (energy carried away to $\infty$).

Strategy: Evaluate

$$S = \sum_a -m_a c \int_a ds - \frac{1}{2c^2}\int d^4x\, j^i A_i$$

where $A_i$ now solves $\partial_i F^{ij} = \frac{4\pi}{c} j^j$

to $\underline{O}((v/c))$ $\longrightarrow$

We can go now to next order in $\frac{v}{c}$ expansion ...

— eqn (85.1) called $L^{(0)}$ — 0-th order in $\frac{v}{c}$

— we can go to $\left(\frac{v}{c}\right)^2$, actually

( — radiation $\sim \frac{1}{c^3}$ — so occurs in next order only ... )

We start w/ Lagrangian for a-th particle

$$L_a = -m_a c^2 \sqrt{1 - \frac{v_a^2}{c^2}} - e_a \phi + \frac{e_a}{c} \vec{v}_a \cdot \vec{A} -$$

— & calculate $\phi$ & $\vec{A}$ at $\vec{x}_a$ induced by all other charges — if we get $\phi$ to $\underline{O}\left(\frac{v^2}{c^2}\right)$ & $\vec{A}$ to $\underline{O}\left(\frac{v}{c}\right)$, we'll have $L_a$ (when we plug them back) to $\underline{O}\left(\frac{v^2}{c^2}\right)$ ...

We use the general expressions for retarded $\phi$ & $\vec{A}$

$$\phi(\vec{x}, t) = \int d^3\vec{x}' \, \frac{\rho_{\not a}\left(\vec{x}', t - \frac{|\vec{x} - \vec{x}'|}{c}\right)}{|\vec{x} - \vec{x}'|}$$

we want this @ $\vec{x}_a$

density of all other charges

since charges are nonrelativistic $\rho$ will not change much for times $\sim \frac{|\vec{x} - \vec{x}'|}{c}$, so we can expand

typical time scale of change of $\rho$ $\sim \frac{(a)}{v}$ [illegible] expanding $\rho$ in $\frac{|\bar{x}-\bar{x}'|}{c}$ this is total charge of system - won't change in time $\rightarrow 0$. (191)

we have

$$\phi(\bar{x},t) = \int d^3x' \frac{\rho(\bar{x}',t)}{|\bar{x}-\bar{x}'|} - \frac{\partial}{\partial t}\left(\int d^3x' \frac{\rho(\bar{x}',t)}{\cancel{|\bar{x}-\bar{x}'|}}\right)\frac{\cancel{|\bar{x}-\bar{x}'|}}{c}$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial t^2}\int d^3x' \frac{\rho(\bar{x}',t)}{\cancel{|\bar{x}-\bar{x}'|}} \frac{|x-x'|^2}{c^2} + \ldots$$

$$\phi(\bar{x},t) \approx \int d^3x' \frac{\rho(\bar{x}',t)}{|\bar{x}-\bar{x}'|} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\left(\int d^3x' |\bar{x}-\bar{x}'| \rho(\bar{x}',t)\right)$$

(191.1)

as for $\bar{A}$, we do the same:

$$\vec{A}(\bar{x},t) = \frac{1}{c}\int \frac{\vec{J}\left(\bar{x}', t - \frac{|\bar{x}-\bar{x}'|}{c}\right)}{|\bar{x}-\bar{x}'|} d^3x' =$$

$$\approx \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}',t)}{|\bar{x}-\bar{x}'|}$$

to leading order in $\vec{v}/c$ ($\vec{A}$ has an $O(\frac{v}{c})$ term in $L$ already.)

(191.2)

to make 191.1 & 191.2 manageable, imagine $\rho$ correspond to a single charge at $\bar{x}_0$ $\longrightarrow$

this means $\rho(\vec{x}', t) = q\, \delta^{(3)}(\vec{x}' - \vec{x}_0(t))$

$$\vec{j}(\vec{x}', t) = q\, \dot{\vec{x}}_0(t)\, \delta^{(3)}(\vec{x}' - \vec{x}_0(t))$$

$$= q\, \vec{v}_0(t)\, \delta^{(3)}(\vec{x}' - \vec{x}_0(t))$$

hence:

$$\varphi(\vec{x}, t) = \underbrace{\frac{q}{|\vec{x} - \vec{x}_0(t)|}}_{(191.1)} + \underbrace{\frac{q}{2c^2} \frac{\partial}{\partial t^2} |\vec{x} - \vec{x}_0(t)|}_{\partial_t \left( \frac{\partial}{\partial t} \dots \right)}$$

(191.2)

$$\hookrightarrow \vec{A}(\vec{x}, t) \simeq \frac{q}{c} \frac{\vec{v}_0(t)}{|\vec{x} - \vec{x}_0(t)|}$$

Tricks: (for convenience, really!!) let

$$\phi'(\vec{x}, t) = \phi(\vec{x}, t) - \frac{1}{c} \frac{\partial f(\vec{x}, t)}{\partial t}$$

$$\vec{A}'(\vec{x}, t) = \vec{A}(\vec{x}, t) + \vec{\nabla} f(\vec{x}, t)$$

w/ $f = \dfrac{q}{2c} \dfrac{\partial}{\partial t} |\vec{x} - \vec{x}_0(t)|$

$$\Rightarrow \phi'(\vec{x}, t) = \frac{q}{|\vec{x} - \vec{x}_0(t)|}$$

$$\vec{A}'(\vec{x}, t) = \frac{q\, \vec{v}_0(t)}{|\vec{x} - \vec{x}_0(t)|} + \frac{q}{2c} \vec{\nabla} \frac{\partial}{\partial t} |\vec{x} - \vec{x}_0(t)|$$

$$= \frac{q\, \vec{v}_0(t)}{c\, |\vec{x} - \vec{x}_0(t)|} + \frac{q}{2c} \frac{\partial}{\partial t} \underbrace{\vec{\nabla} |\vec{x} - \vec{x}_0(t)|}_{(\vec{x} - \vec{x}_0(t))/|\vec{x} - \vec{x}_0(t)| = \vec{n}(t)}$$

$$\vec{A}'(\bar{x},t) = \frac{q\,\bar{v}_0(t)}{c\,|\bar{x}-\bar{x}_0(t)|} + \frac{q}{2c}\,\dot{\vec{n}}(t)$$

$$\vec{n}(t) = \frac{\bar{x}-\bar{x}_0(t)}{|\bar{x}-\bar{x}_0(t)|}$$ unit vector from $\bar{x}_0(t)$ to $\bar{x}$

$$\dot{\vec{n}}(t) = -\frac{\dot{\vec{x}}_0(t)}{|\bar{x}-\bar{x}_0(t)|} - \frac{\bar{x}-\bar{x}_0(t)}{|\bar{x}-\bar{x}_0(t)|^2}\frac{d}{dt}|\bar{x}-\bar{x}_0(t)| =$$

$$= -\frac{\vec{v}_0(t)}{|\bar{x}-\bar{x}_0(t)|} - \frac{\vec{x}-\vec{x}_0(t)}{|\bar{x}-\bar{x}_0(t)|}\,\frac{(\bar{x}-\bar{x}_0(t))\cdot(-\vec{v}_0(t))}{|\bar{x}-\bar{x}_0(t)|^2}$$

$$\Rightarrow \dot{\vec{n}}(t) = \frac{-\vec{v}_0(t) + \vec{n}(t)\,(\vec{n}\cdot\vec{v}_0(t))}{|\bar{x}-\bar{x}_0(t)|}$$

for many charges — $\Sigma$ their effects

So
$$\begin{cases} \phi'(\bar{x},t) = \dfrac{q}{|\bar{x}-\bar{x}_0(t)|} \\[2ex] \vec{A}'(\bar{x},t) = \dfrac{q\,\vec{v}_0(t) + \vec{n}(t)\,(\bar{n}(t)\cdot\bar{v}_0(t))\,q}{2c\,|\bar{x}-\bar{x}_0(t)|} \end{cases}$$

in
$$L_a = \underbrace{-m_a c^2\sqrt{1-\frac{v_a^2}{c^2}}}_{-m_a c^2 + \frac{1}{2}m_a v_a^2 + \frac{1}{8}\frac{m_a v_a^4}{c^2}} - q_a\,\phi' + \frac{q_a}{c}\,\bar{v}_a\cdot\vec{A}'$$

equivalently, one can

expand our eqn on top of p 4 ..

but what follows is easier!

go to 190 bottom

follow

to 193 bottom

we have $A_0'(x,t) = \sum_a \frac{q_a}{|x - x_a(t)|}$

$$\vec{A}'(x,t) = \sum_a \frac{q_a\left(\vec{v}_a(t) + \vec{n}_a(t)\,(\vec{n}_a(t)\cdot\vec{v}_a(t))\right)}{2c\,|\bar{x} - \bar{x}_a(t)|}$$

$$\bar{n}_a(t) = \frac{\bar{x} - \bar{x}_a(t)}{|\bar{x} - \bar{x}_a(t)|}$$

$$-\frac{1}{2c^2}\int d^4x\, j^i A_i = -\frac{1}{2}\int dt\, d^3x\, \rho A_0'$$

$$+\frac{1}{2c}\int dt\, d^3x\, \vec{j}(x,t)\cdot\vec{A}'(\bar{x},t) =$$

$$= -\frac{1}{2}\int dt \int \sum_b q_b\, \delta^{(3)}(x - x_b(t))\, A_0(x,t)\, d^3x$$

$$+\frac{1}{2c}\int dt \int \sum_b q_b\, \delta^{(3)}(\bar{x} - \bar{x}_b(t))\, \vec{v}_b(t)\cdot\vec{A}'(\bar{x},t)\, d^3x$$

$$= -\frac{1}{2}\int dt \sum_a \sum_b \frac{q_a q_b}{|x_a(t) - x_b(t)|}$$

$$+ \frac{1}{2c}\int dt \sum_a \sum_b \frac{q_a q_b \left(\vec{v}_b(t)\cdot\vec{v}_a(t) + (\vec{v}_b\cdot\vec{n}_{ab})(\vec{v}_a\cdot\vec{n}_{ab})\right)}{2c\,|x_b(t) - x_a(t)|}$$

drop $\int dt$ to write $\mathcal{L}$

drop $\sum a = b$, use $\sum_{a,b,\,a\neq b} = 2\sum_{a>b}$

$$\mathcal{L} = \sum_a\left(\frac{m_a \vec{v}_a^{\,2}}{2} + \frac{1}{8}\frac{m_a v_a^4}{c^2}\right)$$

$$- \sum_{a\neq b} \frac{q_a q_b}{|x_a(t) - x_b(t)|}$$

$$+ \sum_{a\neq c} \frac{1}{c^2 2} \frac{\bar{v}_a\cdot\bar{v}_b + (\bar{v}_b \bar{n}_{ab})(\bar{v}_a\cdot\bar{n}_{ab})}{|x_b(t) - x_a(t)|}$$

still a nice $\mathcal{L}$, all fields @ same time.

"Darwin $\mathcal{L}$" (famous)

from (191)

$$\phi(\vec{x},t) = \int d^3x' \frac{\rho(x',t)}{|x-x'|} + \frac{1}{2}\frac{\partial^2}{\partial t^2}\int d^3x' \frac{\rho(x',t)}{|x-x'|}\frac{|x-x'|^2}{c^2}$$

$$- \frac{1}{6}\frac{\partial^3}{\partial t^3}\int d^3x' \frac{\rho(x',t)}{|x-x'|}\frac{|x-x'|^3}{c^3}$$

$$= \underbrace{\phi^{(0)} + \phi^{(2)}}_{\text{already discussed in } \mathcal{O}(\frac{v^2}{c^2}) \text{ Lagrangian}} + \phi^{(3)}$$

$$\phi^{(3)} = \frac{\partial}{\partial t}\left[-\frac{1}{6c^3}\frac{\partial^2}{\partial t^2}\int d^3x' \, |x-x'|^2 \rho(x',t)\right]$$

$$\vec{A}(\bar{x},t) = \underbrace{\frac{1}{c}\int d^3x' \frac{\vec{j}(\bar{x}',t)}{|x-x'|}}_{\vec{A}^{(1)}} \underbrace{- \frac{1}{c^2}\frac{\partial}{\partial t}\int d^3x' \, \vec{j}(\bar{x}',t)}_{\vec{A}^{(2)}} + \underbrace{\frac{1}{c^3}\frac{\partial^2}{\partial t^2}\left(\int d^3x' \, |x-x'| \, \vec{j}(\bar{x}',t)\right)}_{\vec{A}^{(3)}}$$

$\vec{A}^{(1)}$: already discussed in $\mathcal{O}(\frac{v^2}{c^2})$ Lagrangian

$\vec{A}^{(2)}$: recall

$$\int d^3x' \, \partial_\alpha \vec{x}' \cdot j^\alpha(x',t) =$$

$$= \int d^3x' \, \partial_\alpha(\vec{x}' j^\alpha) - \int d^3x' \, \vec{x}' \partial_\alpha j^\alpha =$$

$$\oint_{S^2_\infty} \vec{x}' j^\alpha(x',t) d^3x' = 0$$

(no currents @ $\infty$)

$$= + \int d^3x' \, \vec{x}' \frac{\partial}{\partial t}\rho(x',t)$$

$$= \frac{\partial}{\partial t}\int d^3x' \, \vec{x}' \rho(x',t)$$

$\vec{A}^{(3)}$: this we discard, since higher $\mathcal{O}(\frac{v^3}{c^3})$, when plugged into $L$: $\mathcal{O}(\frac{v^4}{c^4})$

hence

$$\vec{A}^{(2)} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\int d^3x' \, \vec{x}' \rho(x',t)\right)$$

$$\phi^{(3)} = \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\left[-\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\left(\int d^3x' \, |x-x'|^2 \rho(x',t)\right)\right]}$$

this begs to be gauge transformed away

So we take $f^{(2)} = -\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \, |x-x'|^2 \rho(x',t)$

$$\phi^{(3)'} = \phi^{(3)} - \frac{1}{c}\frac{\partial f^{(2)}}{\partial t} = 0$$

$$\vec{A}^{(2)'} = \vec{A}^{(2)} + \vec{\nabla} f^{(2)} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \, \vec{x}' \rho(x',t)$$

$$-\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \, \underbrace{\vec{\nabla}_x |x-x'|^2}_{2(\bar{x}-\bar{x}')} \rho(x',t)$$

$$\begin{cases} \vec{A}^{(2)'} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int d^3x' \left(-\frac{1}{3}\bar{x} - \frac{2}{3}\bar{x}'\right)\rho(\bar{x}',t) \\ \phi^{(3)'} = 0 \end{cases}$$

take $\rho(\bar{x}',t) = \sum_b q_b \, \delta^{(3)}(\bar{x}' - \bar{x}_b(t))$

$$\underset{(\bar{x},t)}{\vec{A}^{(2)'}} = \frac{1}{3c^2}\frac{\partial^2}{\partial t^2}\sum_b q_b\left(-\vec{x} - 2\bar{x}_b(t)\right) = -\frac{2}{c^2 3}\sum_b q_b \ddot{\vec{x}}_b(t)$$

no $\vec{x}$-dependence, only $t$!

$$\boxed{\vec{E}^{(2)'} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A}^{(2)'} = \frac{2}{3c^3}\sum_b q_b \dddot{\vec{x}}_b(t) = \frac{2}{c^3 3}\dddot{\vec{d}}}$$

$\left(\vec{B}^{(2)'} = \vec{\nabla}\times\vec{A}^{(2)'} = 0, \text{ no } \bar{x}\text{-dep.}\right)$

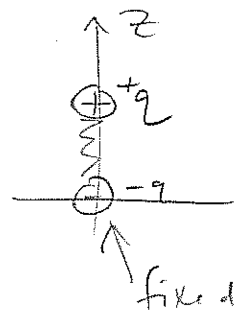
a constant (in $\vec{x}$) $\vec{E}$ field acting on charge ...

( $\equiv$ exactly our rad'n force of p. 198 )

! This derivation emphasizes validity only when small (expansion used)

Nevertheless, in a certain limit (!), we can describe the back reaction of the radiation on the motion of the particle that radiates by a "friction" term $\Rightarrow$ this is called "radiation damping"

Consider our oscillating dipole:




$$m\ddot{z} = -kz \quad , \quad \omega^2 = \frac{k}{m}$$

so

$$m\ddot{z} = -\omega^2 m z$$

the energy it radiates per unit time, averaged over period, is $P = \frac{2e^2}{3c^3}\overline{|\ddot{z}|}^2$ $\quad \overline{|\ddot{z}|} = \omega^2 z_0$

amplitude of oscillations (we called it "a" there)

this means that

$$\frac{d}{dt}\left(\frac{m\dot{z}^2}{2} + \frac{m\omega^2}{2} z\right) \neq 0$$
anymore

but rather, when averaged per period, should give $P$!

Imagine that this is described by adding $f_{rad}$ to the r.h.s of EOM: $m\ddot{z} + \omega^2 m z = f_{rad}$ $\longrightarrow$

follow usual steps in getting $\frac{d}{dt} E = 0$, 

now w/ frad :

$$m\ddot{z} + \omega^2 m z = f_{rad} \quad | \times \dot{z}$$

$$m\ddot{z}\dot{z} + \omega^2 m z \dot{z} = f_{rad}\,\dot{z}$$

$$m \frac{d}{dt}\left(\dot{z}^2 \frac{1}{2}\right) + \omega^2 m \frac{1}{2}\frac{d}{dt}(z^2) = f_{rad}\,\dot{z}$$

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{z}^2 + \frac{\omega^2 m}{2} z^2\right) = f_{rad}\,\dot{z}$$

average both sides over period $\overline{\cdots} = \frac{1}{T}\int_0^T dt \ldots$

$$\overline{\frac{d}{dt} E} = \underbrace{\overline{f_{rad}\,\dot{z}}}$$

energy is not conserved anymore

$f_{rad} = ?$ to get $\overline{f_{rad}\,\dot{z}} = -P$

Claim: $f_{rad} \quad \frac{2e^2}{3c^3}\dddot{z}$

Proof: $\frac{2e^2}{3c^3}\overline{\dddot{z}\,\dot{z}} = \frac{2e^2}{3c^3}\frac{1}{T}\int_0^T \dddot{z}\,\dot{z}\,dt =$

$$= \frac{2e^2}{3c^3}\frac{1}{T}\int_0^T \left[\frac{d}{dt}(\ddot{z}\,\dot{z}) - \ddot{z}\,\ddot{z}\right] dt = -\frac{2e^2}{3c^3}\overline{(\ddot{z})^2}$$

$$= -P$$

So, we have new E.O.M. for our dipole:

$$m\ddot{z} = -m\omega^2 z + \frac{2e^2}{3c^3}\dddot{z}$$

radiation reaction force
(( also follows from $A^{(2)}$
see L&L ))

when is this $\underbrace{\ }_{\text{equation}}$ sensible at all ?

for one thing ... imagine set $\omega^2 = 0$

$$\ddot{z} = \frac{2e^2}{3mc^3}\dddot{z} = \frac{2}{3}\underbrace{\left(\frac{r_e}{c}\right)}\dddot{z}$$

time it takes for light to traverse a distance = = classical radius of electron ($10^{-13}$ cm)

( $r_e \equiv \frac{e^2}{mc^2}$ remember )

this has funny solutions

$z = e^{\alpha t}$
$\ddot{z} = \alpha^2 e^{\alpha t}$
$\dddot{z} = \alpha^3 e^{\alpha t}$

$\rightarrow \quad \alpha = \frac{c}{r_e}$ = inverse of that same time ...

self-accelerating solutions

((clearly not so good...))

the sensible point of view on

$$m\ddot{z} = -m\omega^2 z + \frac{2e^2}{3c^3}\dddot{z}$$

is that it should correctly describe backreaction of radiation on charge when $\dddot{z}$ term is $\ll$ 1st two terms

what are the conditions?

- 3rd term

$$\sim m\frac{r_e}{c}\omega^3 z_0$$

$$\wedge\!\!\wedge$$

- 2nd term
  1st term $\sim m\omega^2 z_0$

NB: also a small-field restriction: if $\omega = \frac{eB}{mc}$, $\frac{\omega r_e}{c} \ll 1$ $\Rightarrow B \ll \frac{m^2c^4}{e^3}$

when $\frac{\omega r_e}{c} \ll 1$

or $\omega \ll \frac{c}{r_e} \sim \frac{10^{10}\,\text{cm}}{10^{-13}\,\text{cm}\,s} \sim 10^{23}\,\frac{1}{s}$

$\left(\begin{array}{c}\text{frequency}\\ \text{of radiation}\end{array}\right) \lll 10^{23}\,Hz$

Now $\hbar \sim 10^{-33}\,J\cdot s \sim 10^{-15}\,eV\cdot s$

$\hbar\omega \lll 10^{23}\frac{1}{s} \times 10^{-15}\,eV\cdot s \sim 10^{8}\,eV \sim 100\,MeV$

while $m_e c^2 \times 2 \sim 10^6\,eV \sim MeV$

$\Rightarrow$ energy of photon becomes $>$ $e^+e^-$ pair rest mass at $\omega$ <u>before</u> back reaction becomes $O(1)$.

– so quantum effects kick in much before classical EM & radiation force

(distance scales ($r_e$))   (time scales ($r_e/c$))

become nonsensical (contradictory)

classical radius of e $\ll$ Compton wavelength of e $\ll$ Bohr radius → distance scales

$r_e \sim \dfrac{e^2}{m_e c^2}$   $\dfrac{\hbar}{m_e c} = \lambda_e$   $a_0$

$10^{-13}$ m   $10^{-10}$ m   $10^{-8}$ m

scales are "hierrarchiral" : $\lambda_e = \dfrac{1}{\alpha} r_e$ , $\underbrace{\alpha = \dfrac{1}{137}}_{\text{fine structure constant}}$

$a_0 = \dfrac{1}{\alpha} \lambda_e$

QED   QM   classical EM

classical   Compton   Bohr → distance scales

$10^{-13}$ m   $10^{-10}$ m   $10^{-8}$ m

"particles" are actually fields, just like EM
– quantum field theory takes over

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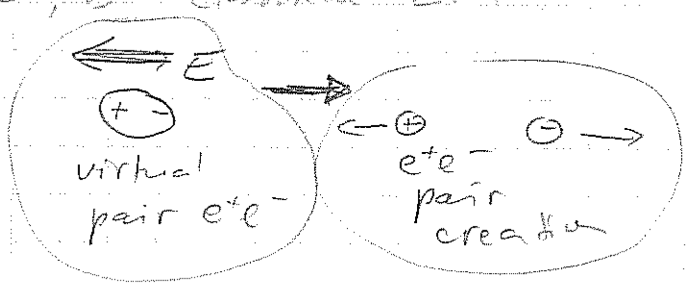
& finally, recall for strong $\vec{E}$, $\vec{B}$ classical EM can also break down ...




$eE_* \cdot \lambda_e \sim 2 m_e c^2$

critical E

$\lambda_e \sim \dfrac{\hbar}{}$

$E_* \sim \dfrac{m_e^2 c^3}{\hbar e}$