We argued that if $s_{12} = 0$, then $s'_{12} = 0$

interval between 1 & 2 (events)

if 1 & 2 are separated by $= 0$ interval in one frame, then interval is $= 0$ in any (primed) inertial frame

Question:

Notation ds^2_{12} simply indicates infinitesimally small interval, i.e., we take

$$(t_1 - t_2)^2 = dt$$

$$(\bar{r}_1 - \bar{r}_2)^2 = d\vec{r}^{\,2}$$

one could put subscripts 12 on dt_{12} & $d\bar{r}_{12}$

but we don't

& we call

$$ds^2_{12} = c^2 dt^2 - d\vec{r}^{\,2}.$$

Question: Then we argued that

$$ds_{12} = 0 \iff ds'_{12} = 0$$

implies that $ds^2_{12} = A\, ds'^2_{12}$

(because if one is zero, so is the other, they have to be $\sim$ to each other; we can also use the linear relation $ds_{12} = a\, ds'_{12}$!)

Why can't $\underline{a}$ depend on $|\bar{r}_1 - \bar{r}_2|$ (or $|t_1 - t_2|$) ?

These are infinitesimal quantities, & so are ds_{12} & ds'_{12}. We argued $a(v'_{12})$ speed! is allowed, now why not $a(v', |\bar{r}_1 - \bar{r}_2|, |t_1 - t_2|)$

(relative speed O, O')

homogeneity in $\bar{r}$ & t & isotropy allow these, after all

But any relation like this means that we have

$$ds_{12} = a(v', |d\bar{r}_{12}|, dt_{12})\, ds'_{12}$$

this is higher order

we are comparing infinitesimals, after all

so $ds_{12} = a(v')\, ds'_{12}$

Question:

We argued

$$ds_{12}^2 = A(v')\, ds_{12}'^2$$

$$ds_{12}'^2 = \tilde{A}(v')\, ds_{12}^2$$

(should be same as function A, since O & O' are on same footing)

Then, we said

$$ds_{12}^2 = A(v')\, ds_{12}'^2 = \underbrace{A(v')^2}_{A(v')^2 = 1} ds_{12}^2$$

but $A(v') = \pm 1$

Qu: why not -1?

consider 3rd inertial system ...

w/ ds''^2_{12}

$$ds''^2_{12} = A(v'')\, ds^2_{12} = A(v'')\, A(v')\, ds'^2_{12}$$

$$\| $$

$$A(v'',')\, ds'^2_{12}$$

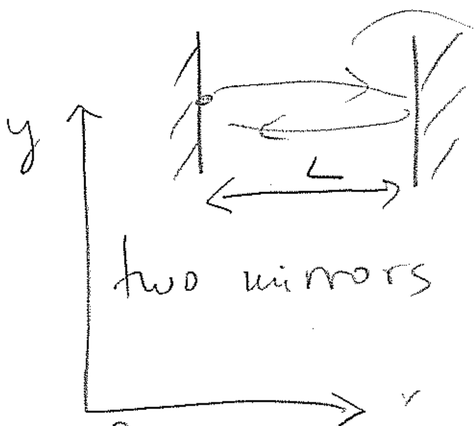
relative speed between O'' & O'

if $A = -1$ we'll have $ds''^2_{12} = + ds'^2_{12}$

$$\underbrace{\begin{matrix} \| \\ -ds'^2_{12} \end{matrix}}_{\text{which doesn't make sense.}}$$

#

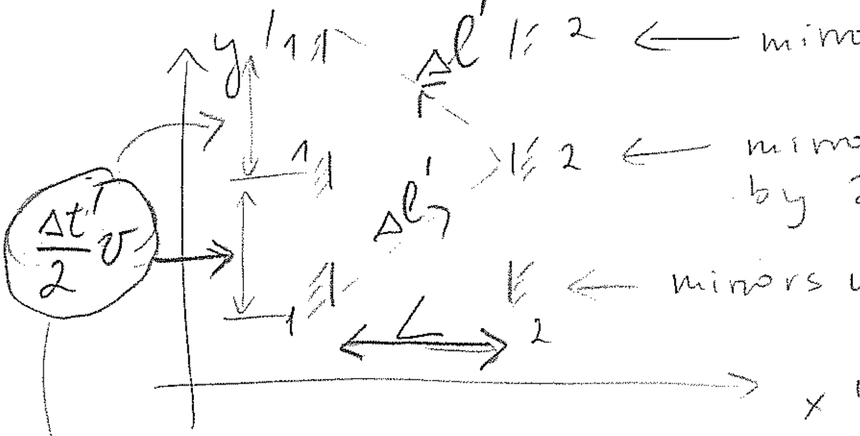
Finally, a more "baby" argument for invariance of (timelike) nonzero interval ...



light pulse bounces back & forth, takes time $\Delta t = 2\frac{L}{c}$

O - @ rest wrt mirrors

now take a frame moving in negative-y direction w/ v; in this frame mirrors move in $+y$ w/ v:



← mirrors when signal back at 1

← mirrors when signal reflected by 2

← mirrors when signal emitted by

because mirrors move w/ v

so light travels along diagonal w/ c:

$$\Delta \ell' = \sqrt{L^2 + v^2 \frac{(\Delta t')^2}{4}}$$

& $\Delta t' = \frac{2\Delta \ell'}{c} = \frac{2}{c}\sqrt{L^2 + v^2 \frac{(\Delta t')^2}{4}}$

$$c^2(\Delta t')^2 = 4L^2 + v^2(\Delta t')^2$$

$\longrightarrow$ solve for $(\Delta t')^2 c^2$

$$c^2(\Delta t')^2 = \frac{4L^2}{1 - \frac{v^2}{c^2}} \qquad (*)$$

interval between emission by 1 & absorption by 1

in O-frame: $\Delta s^2 = c^2 \Delta t^2 = c^2 \frac{4L^2}{c^2} = 4L^2$

(since two events @ same place)

in O'-frame: $\Delta s'^2 = c^2 \Delta t'^2 - \underbrace{\Delta y'^2}_{(\Delta t' v)^2} =$ use $(*)$

(two events separated in y)

$$= \frac{4L^2}{1 - \frac{v^2}{c^2}} - \frac{v^2}{c^2} \underbrace{c^2 \Delta t'^2}_{\frac{4L^2}{1-\frac{v^2}{c^2}}} = \frac{4L^2}{1-\frac{v^2}{c^2}}\left(1 - \frac{v^2}{c^2}\right) = 4L^2$$

hence : $\Delta s^2 = \Delta s'^2$

for timelike separated events as well.