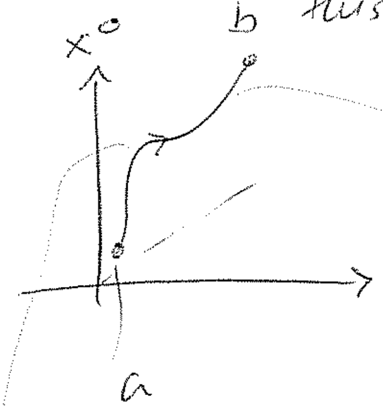
Getting EOM in 4 vector notation from geometric action (directly!):

$$S = -mc \int_a^b ds$$

this is a functional of worldline



parametrize worldline as
$$\begin{cases} x^i = x^i(\theta) \\ \theta \in [0, 1] \\ x^i(0) = a^i \\ x^i(1) = b^i \end{cases}$$

4-coordinates of a & b

(never mind, pretend it's timelike!)

$$ds^2 = dx^i dx_i =$$

$$= \frac{dx^i}{d\theta}\frac{dx_i}{d\theta}(d\theta)^2 \Leftarrow \text{interval along curve.}$$

$$ds = d\theta \sqrt{\frac{dx^i}{d\theta}\frac{dx_i}{d\theta}}$$

makes it clear that S is a functional of $x^i(\theta)$

$$S = -mc \int_{\theta=0}^{\theta=1} d\theta \sqrt{\frac{dx^i}{d\theta}\frac{dx_i}{d\theta}} \equiv S[x]$$

$$\delta S[x] \equiv S[x+\delta x] - S[x] =$$

$$= -mc\int_0^1 d\theta \left\{ \underbrace{\sqrt{\frac{dx^i}{d\theta}\frac{dx_i}{d\theta} + 2\frac{dx^i}{d\theta}\frac{d\delta x_i}{d\theta}}} - \sqrt{\frac{dx^i}{d\theta}\frac{dx_i}{d\theta}} \right\}$$

$$= -mc\int_0^1 d\theta \left\{ \sqrt{\frac{dx^i}{d\theta}\frac{dx_i}{d\theta}} \underbrace{\left\{ 1 + 2\,\frac{\frac{dx^i}{d\theta}\frac{d\delta x_i}{d\theta}}{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}} \right\}^{1/2}}_{\simeq 1 + \frac{dx^i/d\theta\,\frac{d\delta x_i}{d\theta}}{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}} - \sqrt{\frac{dx^i}{d\theta}\frac{dx_i}{d\theta}} \right.$$

$$\simeq -mc\int_0^1 d\theta \, \frac{\frac{dx^i}{d\theta}\frac{d\delta x_i}{d\theta}}{\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}} =$$

$$= -mc\int_0^1 d\theta \frac{d}{d\theta}\left(\frac{dx^i/d\theta}{\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}}\,\delta x_i \right)$$

$$+ mc\int_0^1 d\theta\, \delta x^i \frac{d}{d\theta}\left(\frac{\frac{dx_i}{d\theta}}{\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}} \right) =$$

$$= -mc\, \delta x_i \left. \frac{\frac{dx^i}{d\theta}}{\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}} \right|_{\theta=0}^{\theta=1} \longrightarrow \left.\begin{array}{l} \delta x_i = 0 \\ @\ \theta = 1 \\ \&\ \theta = 0 \end{array}\right\} \text{this term vanishes.}$$

$$+ mc \int_0^1 d\theta\, \delta x_i \frac{d}{d\theta}\left(\frac{\frac{dx_i}{d\theta}}{\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}} \right) = \delta S = 0.$$

$\forall\ \delta x_i(\theta)$ by main variational theorem

$\Rightarrow$

$$\frac{d}{d\theta}\left(\frac{\frac{dx^i}{d\theta}}{\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}}} \right) = 0 \qquad (*)$$

So what is this?

$$\sqrt{\frac{dx^j}{d\theta}\frac{dx_j}{d\theta}} = \sqrt{\frac{ds^2}{d\theta^2}} = \frac{ds}{d\theta}$$

$$(*) = \frac{d}{d\theta}\left(\frac{\frac{dx^i}{d\theta}}{\frac{ds}{d\theta}} \right) = 0 \Rightarrow \frac{d}{d\theta}\left(\frac{dx^i}{ds} \right) = 0 \Rightarrow$$

multiply by $\dfrac{1}{\frac{ds}{d\theta}}$

$$\frac{1}{\frac{ds}{d\theta}} \frac{d}{d\theta}\left(\frac{dx^i}{ds}\right) = 0$$

$$\Downarrow$$

$$\frac{d}{ds}\left(\frac{dx^i}{ds}\right) = 0 \quad \to \quad \frac{du^i}{ds} = w^i = 0.$$

- four acceler. $\equiv 0$
- free particle.