A Guide to Solving Simple Ordinary Differential Equations (ODE's)

Brian Frederick Mullins
University of Toronto, Department of Physics

I. Motivation

For many of you, this may be your first encounter with differential equations. The ability to derive and solve differential equations is essential in most scientific and many non-scientific fields. There are many classes of these equations that describe a wide variety of physical phenomena ranging from the vibrating modes on a drumhead to the quantum mechanical description of an electron in a periodic potential. In our discussion, we will concentrate on learning how to solve a very specific and simple differential equation.

II. Bridging the Gap

At first, the term "*differential equation*" may sound intimidating. However, be assured that differential equations are very similar to equations you have already encountered in your high school math courses such as the quadratic equation. There are two main differences between differential equations and the equations you are already familiar with. The first and most noticeable difference is that differential equations involve taking the *derivative* of a variable instead of a *power*. For comparison:

Quadratic Equation: $ax^2 + bx + c = 0$

Differential Equation: $a\dfrac{dx}{dt} + bx + c = 0$

The second difference is subtler: In the quadratic case, the variable we are solving for is a *number*, while in the differential case, we are solving for a *function*. If you have already taken calculus, you might have guessed this since you can only take the derivative of a function[1]. Let's look at an example:

Suppose we want to solve the following differential equation:

$$\frac{d^2x}{dt^2} + 4x = 0. \qquad (1)$$

[1] A constant (number) is considered a function too!

We know this is a differential equation because it involves a derivative (in this case it is the second derivative of *x* with respect to *t*). In order to make this problem easier to read, let's isolate the derivate term by rewriting equation (1) as:

$$\frac{d^2x}{dt^2} = -4x \qquad (2)$$

Recognize that the variable *x* is a function of *t* since we a differentiating with respect to *t*. Now, by looking at equation (2) we can ask: What function *x(t)* can be differentiated twice to equal the same function multiplied by constant -4? Let's randomly guess a solution $x(t) = t^3$. The second derivative of $t^3$is $6t$. Substituting $x = t^3$ and $\frac{d^2x}{dt^2} = 6t$ into our differential equation (1) gives:

$$6t + 4t^3 \neq 0 \qquad (3)$$

Equation (3) shows that our guess $x(t) = t^3$ is not a solution to equation (1) (and hence not a solution to equation (2)) since the equality in equation (3) is not satisfied[2]. Let's guess again by trying the function $x(t) = \cos(2t)$. The second derivative of $\cos(2t)$ is $-4\cos(2t)$. Substituting $x = \cos(2t)$ and $\frac{d^2x}{dt^2} = -4\cos(2t)$ into equation (1) gives:

$$-4\cos(2t) + 4\cos(2t) = 0 \qquad (4)$$

We find that the equality in equation (4) is satisfied. Therefore, $x(t) = \cos(2t)$ is a solution to the differential equation (1).

III. Solving Simple ODE's – "The Standard Lawn Mower"

Hopefully, you now have a better feel for differential equations. In the example above, we just guessed some functions and tried to see if they were solutions by simply substituting them directly into the differential equation (the first time we were not successful, but the second time we were). Now let's be honest with ourselves – most of us aren't very good guessers, so we need a technique that will give us the right function without some divine revelation. The following description is a mechanical procedure (i.e., no thinking involved – much like using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ when solving quadratic equations) that will solve differential equations of the following form:

$$\frac{dx}{dt} - bf(x)g(t) = 0$$

[2] The term on the left side of the "=" sign does not equal the term on the right side of the "=" sign.

In this equation, x is a function of t, $f(x)$ is some function of x, $g(t)$ is some function of t and b is a constant. No attempt at mathematical justification will be given for this technique except to show though example that it works. We will demonstrate this procedure with the following example:

In the equation above, let $x = V$, $f(x) = V$, $g(t) = 1$ and $b = 6$, where V is some function of t. Therefore, our example differential equation is:

$$\frac{dV}{dt} - 6V = 0 \tag{1}$$

Step (1) **Isolate one term on each side of the equals sign:**

- Add $6V$ to both sides of (1):

$$\frac{dV}{dt} = 6V \tag{2}$$

Step (2) **Split the derivative so that there is only one differential on each side of the equation:**

A differential is a fancy "d" followed by a variable name (i.e., dV). The differential tells you what variable to integrate with respect to (just as the dt in $\frac{dx}{dt}$ tells you to differentiate with respect to t, but there will be more on integration later).

- Multiply both sides of equation (2) by dt:

$$dV = 6Vdt \tag{3}$$

Step (3) **Collect like variables including the differentials on each side of the equation:**

In this case we want all V's and dV's on the left side and all t's and dt's on the right.

- Divide both sides of equation (3) by V:

$$\frac{dV}{V} = 6dt \tag{4}$$

Step (4) **Slap integral signs on both sides of the equation:**

$$\int \frac{dV}{V} = \int 6dt \tag{5}$$

These integrals are called indefinite integrals because they don't have upper and lower bounds above and below the integral signs. There are actually an infinite number of functions (differing only by a constant) that satisfy (is the anti-derivative of) an integral.

Step (5) **Put upper and lower bounds on the integrals:**

$$\int_{Vi}^{Vf} \frac{dV}{V} = \int_{ti}^{tf} 6dt \qquad (6)$$

These integrals are now called definite (or exact) integrals because they contain upper and lower bounds (*Vf*, *tf* and *Vi*, *ti* respectively). The bounds serve to "nail down" one function from the infinite set of possible functions. For each integral, the upper bound contains the final value of the variable that we are integrating with respect to, and the lower bound contains the initial value (also known as the initial condition).

Step (6) **Evaluate the integrals on each side of the equation:**

Ok, some of you may not have integrated before. Don't worry if you don't know how to take the integral (anti-derivative) of a function. Most integrals are too complex and time consuming to solve by hand, so like most people you can just look the integral up in a book[3] or get a computer to solve it.

Evaluating a definite integral involves two steps:

Sub-step (1) **Evaluate the indefinite integral – that is, take the anti-derivative:**

The first indefinite integral on the left hand side of equation (5) is:

$$\int \frac{1}{V} dV = \ln(V) + A$$

where *A* is an arbitrary constant.

The second indefinite integral on the right hand side of equation (5) is:

[3] Tables of indefinite integrals can be found almost anywhere. A good source to start with is at the back of a calculus textbook. The *CRC Handbook of Chemistry and Physics* has good integral coverage and one of the better references is *Shaum's Mathematical Handbook of Formulas and Tables*. Note: Most texts don't include the constant – it is assumed to be there.

$$\int 6dt = 6\int dt = 6t + B$$

where B is an arbitrary constant.

Sub-Step (2) Nail down the exact function from the upper and lower bounds:

There is a theorem called the Fundamental Theorem of Calculus that shows how to evaluate definite integrals:

If $\int f(x)dx = g(x) + C$ **is the indefinite integral where** ***g(x)*** **is the anti-derivative of** ***f(x)*** **and** ***C*** **is an arbitrary constant. Then** $\int_a^b f(x)dx = \left(g(x) + C\right)\Big|_a^b = g(b) - g(a)$ **is the definite integral where** ***b*** **and** ***a*** **are the upper and lower bounds respectively and** ***b*** **>** ***a*****.**

Using the upper and lower bounds on the left hand side of equation (5), the first definite integral evaluates to:

$$\left(\ln(V) + A\right)\Big|_{Vi}^{Vf} = \ln(Vf) - \ln(Vi) \qquad (7)$$

Using the upper and lower bounds on the right hand side of equation (4), the second definite integral evaluates to:

$$\left(6t + B\right)\Big|_{ti}^{tf} = 6tf - 6ti \qquad (8)$$

Step (7) Combine the results:

- Equate equations (7) and (8):

$$\ln(Vf) - \ln(Vi) = 6(tf) - 6(ti) \qquad (9)$$

Step (8) Small celebration:

So there you are. You have just solved a differential equation. Equation (9) is the solution to equation (1)! Unfortunately, the solution may not be in the form you want it, so we need to do some more algebra…

Step (9) Solve for the variable you want to isolate/calculate:

In equation (9), let's say V is velocity and t is time. Now let's say we want to solve for the final velocity Vf as a function of the final time tf assuming we have constant values for the initial conditions Vi and ti. From equation (9):

$$\ln(Vf) = \ln(Vi) + 6(tf) - 6(ti)$$

We need to kill the ln's in order to isolate Vf. Take the exponential of both sides of the equation:

$$e^{\ln(Vf)} = e^{\ln(Vi)+6(tf)-6(ti)} \qquad (10)$$

by the properties of exponential functions:

$$Vf = Vi \cdot e^{6(tf)} e^{-6(ti)}$$

Since Vi and ti are constants, so are their exponentials. Therefore we can write:

$$Vf = Ae^{6(tf)} \qquad (11)$$

$$\text{where } A = Vi \cdot e^{-6(ti)} \qquad (12)$$

Finally, equations (11) and (12) form our solution. Note that Vf in equation (11) is a function of time tf – remember we were solving for a *function*.

Step (10) Check solution by substituting it back into the original differential equation:

In this example, we used the naming conventions Vf and tf to avoid confusion with the integration variables V and t respectively. It may seem confusing at first, but V and t are called "dummy variables" (i.e., they can be named anything like τ or ψ) whose only purpose is to perform the integration. It is the upper and lower bounds that give the final solution its variable name. Since we are solving for the final velocity in terms of time, we can arbitrarily rename $Vf = V(t)$ and $tf = t$. Equation (11) then becomes:

$$V(t) = Ae^{6(t)} \qquad (13)$$

$$\text{where } A = Vi \cdot e^{-6(ti)}$$

and Vi and ti are the initial conditions

- Substitute $V = Ae^{6(t)}$ and $\frac{dV}{dt} = 6Ae^{6(t)}$ into equation (1)

  $$6Ae^{6(t)} - 6Ae^{6(t)} = 0 \quad (14)$$

Since the equality in equation (14) is satisfied, equation (13) is indeed a solution to our original differential equation (1).

IV. Final Remarks

The process of solving differential equations may seem complicated at first, but with a bit of practice you'll soon find it's very easy. Hopefully you will realize that the real power of calculus is unleashed when used to solve differential equations.