1997-1998 Physics Olympiad Preparation Program

— University of Toronto —

Problem Set 4: Optics and Waves

Due February 13, 1998

1) Reflecting lightly

a) To reduce the light reflected by a glass surface, such as a camera lens, the surface is coated with a thin layer of another material which has an index of refraction smaller than that of glass. This results in *two* surfaces: the surface of the coated layer, and the surface between the layer and the glass. What is the least thickness of a layer of refraction index $n = 4/3$, to insure that light falling perpendicularly on the surface and having wavelengths $\lambda_1 = 700$ nm and $\lambda_2 = 420$ nm will be minimally reflected for *both* wavelengths?

b) Canadian $20 and $50 bills (and probably others, soon) have a patch in the corner which is actually a dielectric coating — a coating like the one above. Examining such a bill — especially if you can lay your hands on a brand-new one — estimate the product $n \bullet d$ of the index of refraction of the coating and its thickness. If $n = 1.67$, what is the coating thickness? *[Lev]*

2) A sequel — 'Twister: Part Moo'

Doppler anemometry uses the frequency shift of reflected radar beams to measure windspeed. Find the windspeed of a tornado, using a portable radar which radiates radio waves of frequency $f_o = 10^{10}$ Hz, if the measured frequency shift of the reflected signal f is $\Delta f = f - f_o = 5000$ Hz.

Cow ane-moo-metry uses the frequency shift of Clarabelle, unhappy to be lofted into a twister. If Clarabelle's anxious lowing is two octaves higher than normal, at around A-1760 Hz, at what speed must the twister twist in order that Clarabelle's moo is outside the range of human hearing? Take it that the tortured hearing of a tornado expert is 40 Hz – 14 kHz. *[Lev]*

3) A complete waste of paint...

In the previous problem set, Nipun introduced the simple harmonic oscillator. Let's play around with it!

Olga wants to paint the ceiling of her bedroom but doesn't want to do it by hand. At the local bargain store, she buys an ACME F.S.P.M. (Funky Shape Painting Machine — as seen on TV!). The machine works like this. No matter where the spring-mounted sprayer nozzle is, it experiences a total vector force of $\vec{F} = -k_x\, x\, \hat{x} - k_y\, y\, \hat{y}$ where $\hat{x}$, $\hat{y}$ are unit vectors along x and y (which you effectively set by rotating the machine). On the controller, you can set the spring constants k_x and k_y. By moving the initial point of the machine, you change where x=0 and y=0 is in reference to the room.

a) Olga sets $k_x = k_y$ and hits the ON button. Nothing happens but the paint comes out: the paint sprayer sits directly above the machine. "Hmmm" she says, "I'll have to help it." She pulls the spray nozzle to one side, along the x direction, and releases it. What is the resulting paint pattern? (Sketch the pattern on an x–y graph.)

b) "That's pretty boring". Olga catches the brush, pulls it out in both the x and y direction and releases it. What is painted on her ceiling?

c) "Hey, let's see what this thingy can do!" Olga stops the sprayer, and switches the constants so that $k_x = 4\, k_y$. She pulls the spray-head out so that x=y and releases it. What is the resulting paint pattern?

d) Resetting $k_x = k_y$, Olga decides to try something new. Pulling the sprayer out along the x direction, she gives it an initial push in the y direction. What is the resulting pattern?

e) Olga loves this thing! She does the same thing as in (d) but after she resets the controller so that $k_x = 9k_y$. The pattern please?

f) "This is all fun and everything, but I bought this baby to paint my ceiling." What can Olga do so that the machine will paint every part of her ceiling (make no assumptions about the size of the paint spray pattern, or the amount of paint wasted)? *[James]*

4) Acousto-optic modelocking of laser modes

With so many different-frequency waves being amplified in a laser (see General, Q 3 (b)), the output light typically can be a jumble. This happens because a standing wave of a given frequency can *start up at any time*, so there are many waves of the same frequency but different phase, cancelling one another. Sometimes another element is added to a laser cavity: a mode-locker. This mode locker works like a shutter — when it closes, it kills any mode by attenuating its amplitude. The modes that survive are those which just happened to be going through their zero anyway when the shutter closed, so they lose no amplitude. (To be more accurate, those modes which have such a zero-crossing are interfered with very little, while all others are heavily suppressed.)

For an *acousto-optic (A/O) modelocker*, the shutter is actually more like a railroad switch for light coming to it, sometimes sending light straight on, sometimes sending it off to one side. It consists of an ultrasonic transducer (think of it as a very high frequency speaker) mounted to a block of glass. Ultra-high frequency sound waves are launched in the block of glass, and set up a standing-wave pattern (see General, Q 3a). This A/O modelocker is mounted transverse to the laser axis, so the standing wave is across the beam.

Such a sound wave is a wave of density change, both compression and rarefaction. Compressing glass increases its index of refraction (the *Debye-Sears effect*) so any ultrasonic wave makes a sort of mirage. The standing wave in the glass actually causes alternating parallel planes of higher and lower index, also oscillating in time, with the laser trying to propagate through these planes edge-on. When the standing wave is at its highest amplitude, these make a diffraction grating which scatters the laser modes away; when the standing wave passes through its null point, the index is temporarily flat across the glass (just as for water waves or a guitar string) and the diffraction grating turns 'off'.

a) What frequency should be used to drive a modelocker which has been set at the middle of a laser like this:

laser wavelength	$\lambda = 1\ \mu$m
resonator length	$L = 80$ cm

b) The block of glass for the modelocker is 2cm × 1cm × 1cm, and the ultrasonic wave is launched down the 2cm length. What is the wavelength of the wave in glass, for the frequency found above? Does this make a standing wave over the 2cm length? If not, what might one do to make it right?

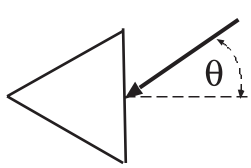
speech of sound in glass (fused silica) $v = 5.958 \times 10^{3}$ m s^{-1}

c) When the laser light passes through the modelocker, at what angle(s) are the rejected modes deflected? *[Robin]*

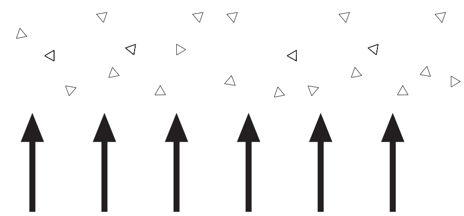
5) Halos and sun-dogs

Ice particles created in the atmosphere form regular hexagonal crystals; they can be long like 6-sided pencils, or flat like a 6-sided plate. Light from the sun scatters through these crystals to make interesting effects in the sky, according to the angle that a light beam is deviated. Since a regular hexagon can be made from an equilateral triangle by lopping the corners off, these ice crystals have a lot in common with prisms you may be familiar with.

a) Take a standard glass prism (flat on the top and bottom, 60° vertices, index of refraction n=1.5). Shine a beam of light (frequency f) on it, as in the diagram. By what angle δ is the beam deviated, as a function of angle θ from surface-normal? Ignore any displacement of the beam and any beam that undergoes total internal reflection (you will see why we can neglect these two later).



b) In the mail, you receive a notice saying "CONGRATULATIONS!!!!!! If you respond to this letter within 24 hours you will receive our GRAND PRIZE!!!". You phone the number given and the next week a package arrives for you. Your grand prize is 50 tiny glass prisms!



What do you do with your cool crystal collection? Some physics of course! You lie all the prisms flat on their ends on a table and spread them apart. From the side, you shine a beam of light on them and examine the resulting pattern on a far wall. You notice that most of the light goes straight through (due to the low concentration of crystals) causing a bright spot on the wall. You also notice that there appear two other bright spots on the wall. Why? — and at what angles relatively to the original incident beam do these spots appear?

To figure this out, use your answer from part (a): determine a range of angles over which the beam does *not* undergo total internal reflection; decide on a set angle-increment $\Delta\theta$ and sketch the deviations of the corresponding beams. For example, if you decide on a range of –20° to 20° and an increment of 10°, graph the beam for $\theta = -20°, -10°, 0°, 10°, 20°$. The angle range and increment is up to you. Chose well, and it becomes apparent that the deviated beams will end up producing two brighter regions of light. There is also an analytical solution if you know how to find the local maximum or minimum of a function — either approach is acceptable. At what angle are these spots and how are they related to the index of refraction of the glass prism?

c) Sometimes, usually on a cold, partly cloudy day, you can see a halo around the sun at 22° away from the line toward the sun. Can the process described in part (a) and (b) be responsible for this halo? If so, how (include a sketch of the light rays entering your eye when you observe this effect)? What is the index of the material that is scattering the light and is this consistent with water crystals? Why do you see a full circle and not just two spots? On a calm day, the halo will be concentrated at two spots, parallel to the horizon. Why this orientation? *[James]*

Prof. Robert Greenler, University of Wisconsin, has a terrific book on this subject: 'Rainbows, Halos, and Glories, published by Cambridge University Press.

6) Sound — reading the Riot Act

Sound waves in air travel without *dispersion*: that is, each frequency-component of sound in air travels at the same speed. Since any sound, even the 'bang!' of a firecracker, can be viewed as being made up of some range of frequencies, this constancy makes life much easier. If sound dispersed as it travelled to us, with high frequencies travelling faster, say, think of what an orchestra would sound like!

However, when sound propagates in a very long sewer-tube, or a long, narrow air-duct, things change. As seen in Problem Set #1 (General), sound in a box or tube has modes, since the air cannot oscillate through the walls. This problem will help you figure out how quickly sound can propagate down a tube or duct, according to its 'travelling modes'.

Start most simply with two practically infinite plates, separated by a distance d, and consider plane waves (flat wavefronts) travelling down the gap, at some angle θ to the walls.

a) For a mode to propagate, we require constructive interference between the reflected waves; otherwise all the bouncing waves will together cancel out. What condition(s) does this set on the wave, for different wavelengths λ? How many modes are there in all?

The *wavenumber, k* is defined by $k = 2\pi/\lambda$. In free-space (no waveguide) this is unique:

$$k_o = \frac{2\pi}{\lambda_o} = \frac{2\pi}{c}\nu_o = \frac{\omega_o}{c}$$

where c is the speed of sound, ν represents the frequency of this wave in cycles/second or Hz, and ω_o is the frequency in radians/second or just s^{-1}.

Then in free space a travelling wave of wavelength λ_o and frequency ν_o has the form:

$$\sin(k_o x - \omega_o t)$$

From this, the *phase speed*, or speed of the wavefronts, is $v_\phi = \omega_o / k_o = c$.

b) For a wave travelling down the gap, what condition(s) does (a) imply for a somewhat different k_{mode}, sometimes called β, which will describe propagation down the gap? What is the phase speed now? How does it compare to the sound speed c?

A sound-burst travels like a pulse, at a different speed, called the *group velocity*:

$$v = \frac{d\omega(\beta)}{d\beta}$$

This is the speed that gives the time it takes for you to hear a noise, some distance away.

c) What is the group velocity for free-space propagation? What is the group velocity for each of the modes of the gap between planes? How do the two compare? Explain it in simple terms, if you can.

d) Take a special case: $d = 20$ cm, and three different frequencies: $\nu_1 = 1$ kHz, $\nu_2 = 1.4$ kHz, and $\nu_3 = 1.8$ kHz. How many modes are there for each frequency? If we play these three notes at the same time for exactly 200 ms, approximately what would we hear at a point 150 m away, down the gap? Roughly what would we hear if we make a sudden 'click' noise, which lasts 2.5 ms and has within it all frequencies in the range (1.2 ± 0.2) kHz?

(Useful: $c = 331 \text{ m s}^{-1}$)

e) Can you generalize: what is the new description if, instead of propagating between two parallel plates, the sound propagates down a duct of rectangular cross-section, with width a and height b? *[Robin]*