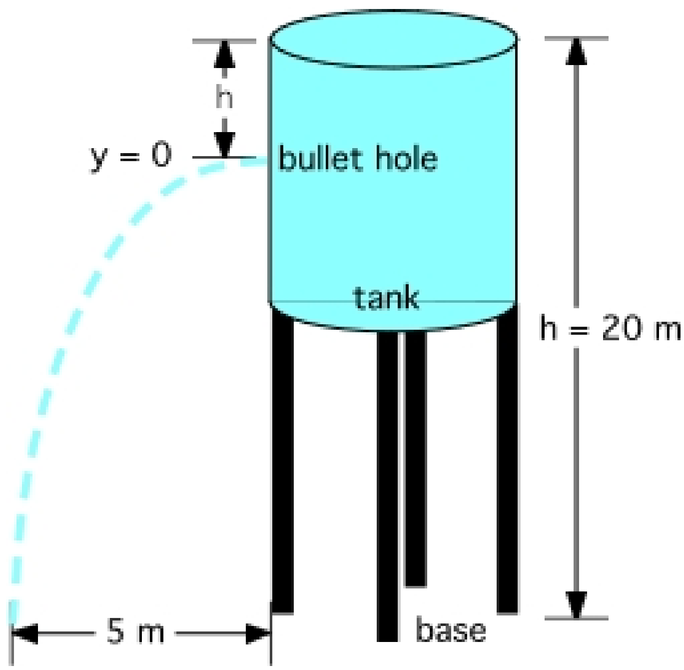
2000–2001 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 2: Mechanics

1) Take your best shot

Due to water pressure in the tank, water squirts out horizontally from the bullet hole at some speed, depending on the 'head' or height of water above where the hole is made. The water then falls a certain vertical distance. The horizontal distance the water reaches depends on the squirt-speed and the time it takes the water to fall — a time that only depends on height above the ground.



Let the hole the rifle makes in the tank be the origin of a vertical axis which we will set to zero. Let h be the height above the hole to the top of the tank.

As water flows out through the bullet-hole, the height of the water in the tank must drop. In a time Δt, the volume of water flowing through the bullet-hole is:

$$A_{flow} \times \text{distance-flowed} = A_{flow} (v_{flow} \times \Delta t)$$

where A_{flow} is the area of the hole, and v_{flow} is the flow-speed of water at the hole.

Likewise the water in the tank will drop by a volume:

$$A_{tank} \times \text{distance-dropped} = A_{tank} (v_{tank} \times \Delta t)$$

Equating these, we get:

$$v_{tank} = v_{flow} \times (A_{flow} / A_{tank})$$

As the water drains from the tank due to the bullet hole, the kinetic energy put into the squirting water simply equals the potential energy of lowering all the water in the tank from the hole up. In a small time Δt this is:

$$\tfrac{1}{2} M_{flow} v_{flow}^{\ 2} = M_{tank}\, g\, h$$

$$\tfrac{1}{2}\,(\rho\, A_{flow}\,(v_{flow}\,\Delta t))\, v_{flow}^{\;2} = (\rho\, A_{tank}\; h)\, g\;\, (v_{tank}\,\Delta t)$$

Where ρ is the density of water. We can cancel terms, and use v_{tank} we found above, to get:

$$v_{flow}^{\;2} = 2\, g\, h$$

This is the initial horizontal speed. The initial vertical speed is zero. The time of flight for the water leaving the tank before it hits the ground 5 meters away from the base of the tank is given by:

$$t^2 = 2\,(20 - h)/g$$

This time needs to be exactly the value that lets the water squirt 5 m sideways, at constant speed v_{flow}, before hitting the ground.

$$v_{flow}\;\, t = 5\ \text{m}$$

therefore,

$$4\, h\, (20 - h) = 25$$

Solving for h we get the result that in order for the water to reach the athlete 5 m away from the base of the tank, ***she must shoot a hole approximately 32 cm down from the top of the tank.*** *[Sal]*

2) The fountain of (misspent) youth

Assume that Suku is at height H. The speed of the water at this height, *before* the collision with the garbage can, is

$$v^2 = v_0^{\;2} - 2\, g\, H$$

where $v_0 = 20\ \text{m s}^{-1}$ is the initial speed. After the collision, the water stops its motion, since the collision between a liquid and any surface is always inelastic. The change in momentum of the water during the collision acts provides the a force and compensates for gravity in order to hold the garbage can and Suku up. This force is equal to:

$$F = \Delta m * (v - 0)/\Delta t = 150\ (\text{kg/s}) * v$$

where Δm is the mass of the water which has a collision at a certain time. Δt is the collision time. Both Δm and Δt are supposed to be infinitessimals (vanishingly small numbers) but their *ratio* is actually equal to the rate of the water flow, which is 150 kg/s. This force is equal to the weight of the garbage can plus Suku.

$$F = (60 + 2.5) \bullet 9.8 = 150\, v = 150 \bullet \sqrt{400 - 2 \bullet 9.8 \bullet H}$$

From which we can find ***H* = *19.5 m*.** Yikes! *[Yaser]*

3) Huh? Hmmm…

a) The water pressure at the bottom of all the containers is the same and equal to (ρ*g* H) where ρ is the density of the water. The pressure is dependent on the height of the water and not on the amount of the water. The total force which is exerted by the water on the bottom plate, by this pressure, is equal to the product of the pressure and area of the plate and is the same for all.

b) The weigh-balance shows three different values for the containers full of water. For the non-cylindrical containers, this value is *not* equal to the product of the water pressure and the bottom-surface area. For the cylindrical container, the water pressure on the side walls is everywhere *sideways*, and contributes nothing to the weight. This isn't true for the other containers.

For the tapered containers, not only water pressure exerts a force over the bottom plate: the rigid *walls of the container* also exert force. For the one in which the area of the top is bigger, this force is downward and therefore the total force on the bottom plate is *more* than the product of pressure and area. For the other container, in which the area of the top is smaller, this force is upward, and therefore the total force is *less* than than the product of the pressure and the area.

The easiest way to find the right answer is to imagine the sloped wall in a series of stair-steps, and then take the limit of the steps becoming vanishingly tiny.

Each step has a vertical part and a horizontal part. Pressure on the vertical part, which is a tiny cylinder, gives an outward force everywhere; these radial forces add up to zero net force. Pressure on the horizontal part, like a shelf around the bottom of the little cylinder, gives a net downward force, and these side-wall forces add up to exactly the weight of the water which lies above the sloped walls (i.e., outside the cylinder lying above the circular base).

The net force can be found by adding all the contributions. Each horizontal step is actually an *annulus* — a flat ring lying between two circles of radius r and $r + dr$.

For a wall sloped at an angle θ from the vertical, the radius r of the vessel as a function of vertical position z from the base is:

$$r = r_0 + z \cdot \tan\theta$$

It's easy to find dr then (as the *differential* of r):

$$dr = dz \cdot \tan\theta$$

The area of the annulus is then:

$$A = 2\pi r \cdot dr = 2\pi r \cdot dz \cdot \tan\theta$$
$$= 2\pi(r_0 + z\tan\theta) \cdot dz \cdot \tan\theta$$

The downward force on this small annulus is then:

$$dF_y = p(z) \cdot A$$

where *p(z)* is the pressure, which is a function of z because z gives the depth. The pressure is simply the weight per unit area of all the water lying above, and so

$$p(z) = \rho g (H - z)$$

Thus

$$\begin{aligned} dF_y &= p(z) \cdot A \\ &= \rho g (H - z) \cdot 2\pi \left(r_o + z \tan\theta \right) \cdot dz \cdot \tan\theta \end{aligned}$$

The total force down on the side walls is then the sum of all these tiny contributions, for all values of z from z=0 to z=H. This is just the integral (siimpler than it seems — the integrand is only a polynomial, actually):

$$\begin{aligned} F_y &= \int_{z=0}^{z=H} dF_y \\ &= \int_{z=0}^{z=H} \rho g (H - z) \cdot 2\pi \left(r_o + z \tan\theta \right) \cdot dz \cdot \tan\theta \\ &= 2\pi\rho g \tan\theta \int_{z=0}^{z=H} (H - z) \cdot \left(r_o + z \tan\theta \right) \cdot dz \\ &= 2\pi\rho g \tan\theta \int_{z=0}^{z=H} \left\{ H r_o + \left(H \tan\theta - r_o \right) \cdot z - \tan\theta \cdot z^2 \right\} \cdot dz \\ &= 2\pi\rho g \tan\theta \left\{ H r_o z \Big|_0^H + \tfrac{1}{2} \left(H \tan\theta - r_o \right) \cdot z^2 \Big|_0^H - \tfrac{1}{3} \tan\theta \cdot z^3 \Big|_0^H \right\} \\ &= 2\pi\rho g \tan\theta \left\{ \tfrac{1}{2} H^2 r_o + \tfrac{1}{6} H^3 \tan\theta \cdot \right\} \end{aligned}$$

Naturally, as θ goes to zero (as for a cylinder), this vertical wall-force contribution goes to zero.

It is obvious that this down-force on the sidewalls is the same as the weight of the water in the 'extra' region outside the cylinder — going back to the term for pressure, all we have done is simply to add the weight of the water lying above every little horizontal annular step. Since we considered every little horizontal step, we've considered the whole volume of water in the extra region. *[Yaser & Robin]*

4) All aboard!

a) Assume the train experiences constant acceleration (deceleration).

Known quantities converted to SI units where appropriate:

Variable Symbol	Description	Numerical Value
$X(0)$	Initial displacement of train	0.00 m
$X(t)$	Final displacement of train	8000 m
$V(0)$	Initial velocity of train	65.0 m/s
$V(t)$	Final velocity of train	0.00 m/s

Unknown quantities:

Variable Symbol	Description
μ	Coefficient of rolling friction

the train experiences a rolling frictional force (proportional to its normal force) that is opposite to its direction of motion:

$$F = -\mu mg$$

$$F = ma$$

$$ma = -\mu mg$$

$$\mu = -\frac{a}{g} \qquad (1)$$

constant acceleration displacement equation:

$$X(t) = X(0) + V(0)t + \frac{1}{2}at^2$$

since $X(0) = 0.00$ m, then

$$X(t) = V(0)t + \frac{1}{2}at^2 \qquad (2)$$

constant acceleration velocity equation:

$$V(t) = V(0) + at$$

since $v(t) = 0.00$ m/s, then

$$V(0) + at = 0$$

$$t = \frac{-V(0)}{a} \qquad (3)$$

substitute equation (3) into equation (2) to find the train's constant acceleration:

$$X(t) = -\frac{V(0)^2}{a} + \frac{V(0)^2}{2a}$$

$$2aX(t) = V(0)^2 - 2V(0)^2$$

$$a = -\frac{V(0)^2}{2X(t)} \qquad (4)$$

substitute equation (4) into equation (1) to find the coefficient of rolling friction:

$$\mu = \frac{V(0)^2}{2gX(t)} = \frac{(65.0)^2}{2(9.81)(8000)}$$

$$\boxed{\mu = 0.0269}$$

SOLUTION FOR PART (B):

Note that although the rolling frictional acceleration μg is constant, since $F = \mu g m$, the rolling frictional force will not be constant if the train's mass is changing – that is: $F(t) = \mu g m(t)$.

Known quantities:

Variable Symbol	**Description**	**Numerical Value**
V	Constant velocity of train	1.00 m/s
m	Initial mass of train and hopper car	75000 kg
l	Length of hopper car	15.0 m
r	Mass transfer rate of grain	1000 kg/s
μ	Coefficient of rolling friction	0.0269

Unknown quantities:

Variable Symbol	**Description**
$m(t)$	Mass of train and hopper car as a function of time
$m(x)$	Mass of train and hopper car as a function of displacement
W	Work done by the train

mass of train and hopper car as a function of time:

$$m(t) = m + r\,t$$

convert *m(t)* to *m(x)* using the fact that $t = \frac{x}{V}$:

$$m(x) = m + r\frac{x}{V}$$

in order to maintain a constant velocity the train must exert a force that balances the rolling frictional force:

$$\sum F = F_{friction} + F_{train} = 0$$

$$F_{train} = -F_{friction} = -(-\mu\, mg) = \mu\, mg$$

since *m* is a function of displacement:

$$F(x) = F_{train} = \mu\, g\, m(x) = \mu\, g\left(m + r\frac{x}{V}\right)$$

the definition of work is:

$$W = \int_0^l F(x)dx$$

$$W = \mu g\int_0^l m(x)dx = \mu g\int_0^l \left(m + r\frac{x}{V}\right)dx$$

$$W = \mu g\left(mx + r\frac{x^2}{2V}\right)\Bigg|_0^l$$

$$W = \mu g\left(ml + r\frac{l^2}{2V}\right) = (0.0269)(9.81)\left((75000)(15.0) + (1000)\frac{(15.0)^2}{2(1.00)}\right)$$

$$\boxed{W = 3.27\times 10^5\ \text{J}}$$

[Brian]

5) Java jump-up (eeyow!)

Given that the weight of the mug is M and that the weight of the bottom of the mug is $\frac{M}{4}$ the height of the centre of mass ($y_{c.m.}$) of the mug with coffee of density ρ (~1 g cm^{-3}) filled to height h is given by,

$$y_{c.m.} = \frac{\left(\frac{3}{4}\frac{MH}{2} + \rho\pi R^2\frac{h^2}{2}\right)}{(M + \rho\pi R^2 h)}$$

Define a constant $a = \rho\pi R^2$ such that the mass of the coffee contained is $m = \rho\pi R^2 h \equiv a h$

$$\therefore\ y_{c.m.} = \frac{\left(\frac{3}{8}MH + \frac{ah^2}{2}\right)}{(M + ah)}$$

The maximum stability of the mug of coffee is achieved for the lowest centre of mass. To find the minimum of $y_{c.m.}$ we need

$$\frac{d(y_{c.m.})}{dh} = \frac{ah(M + ah) - (a)\left(\frac{3}{8}MH + \frac{ah^2}{2}\right)}{(M + ah)^2} = 0$$

$$\therefore\ h(M + ah) = \frac{3}{8}MH + \frac{ah^2}{2}$$

$$ah^2 + Mh - \frac{3}{8}MH = 0$$

Solving for h we get

$$h = \frac{1}{a}\left(-M \pm \left(M^2 + \frac{3}{4}MHa\right)^{\frac{1}{2}}\right)$$

the negative solution can be ignored.

$$h = \frac{1}{a}\left(-M + \left(M^2 + \frac{3}{4}MHa\right)^{\frac{1}{2}}\right)$$

Given:

$M = 200$ g, $\quad H = 20$ cm $\quad a = \rho\pi R^2 = 36\pi\, g/cm$

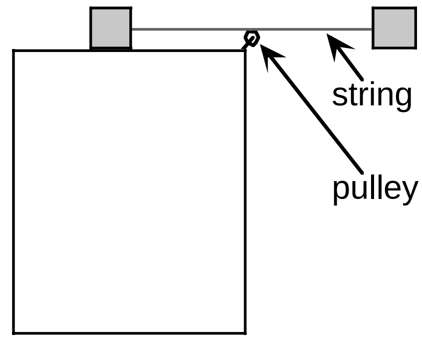
Solving for h we get,

$$h = \frac{1}{36\pi}\left(-200 + \left(40000 + \frac{3}{4}(200)(20)(36\pi)\right)\right)^{\frac{1}{2}}$$

$$h = 3.6\,cm$$

The mug is most stable with coffee filled to a level h = 3.6 cm. *[Sal]*

6) Weight! Bob!



In an experiment, the table-based weight hits the pulley before the free one smacks into the side of the table.

How to explain this?

- It's easiest to analyse the forces in x (horizontal) and y (vertical) components.

- The force on either block comes from tension in the string

- There's only one tension, so the forces on the block are equal in magnitude, though they may change in time

- The difference is that the force on the left-hand block is always directed along the string, towards the pulley. The force on the right-hand block is also always directed along the string, but this vector changes as the angle of the string changes.

- The left-hand block moves horizontally towards the pulley with an acceleration:

  $a_{left} = T/m$ where m is the mass of the block, and T is the tension in the string

- The right-hand block moves horizontally towards the wall with an acceleration:

  $a_{right} = T\cos(\theta)\ /m$

where m: mass of the block, T: tension in the string, θ: the angle the string makes with the horizontal

- Before the right-hand block falls, there is no tension in the string. Once it does begin to move, then it makes an angle $\theta > 0$, so $a_{right} < a_{left}$

- The left-hand block is always accelerated to its collision with the pulley faster than the right hand block to its collision with the wall.

The left-hand block hits first. *[Yaser, Robin]*