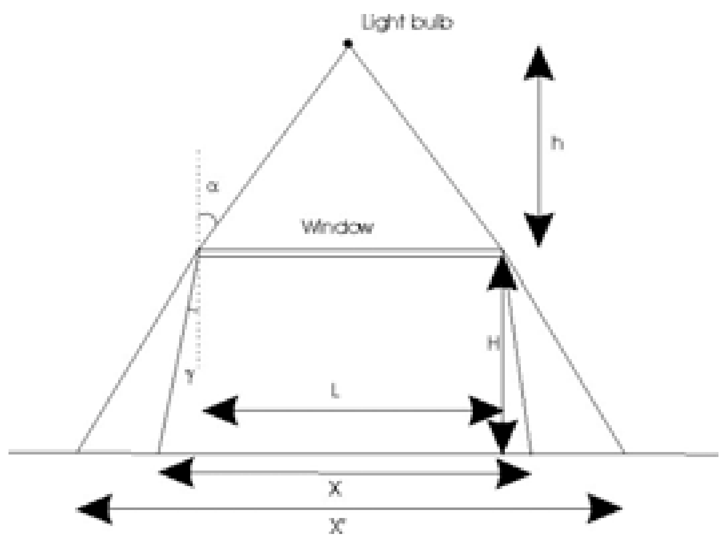
2000-2001 Physics Olympiad Preparation Program

— University of Toronto —

Solution Set 4: Optics and Waves

1) Sub hubbub

This question is supposed to be the easiest one in the set. If you want to consider the size of the window and the light bulb, you'll find the geometry a bit fussy without teaching very much. The reasonable approximation is that the light bulb is tiny compared to other sizes, like the distance from window and therefore its size should be neglected. A thick window also can be solved as we do below (with an extra refraction surface), but let's take the window to be thin compared to other distances, like h and the height H from the bottom of the sea.



The area at the bottom of the sea which is illuminated, is square in shape, each side is equal to an unknown, X. The angles that the rays traveling in the sub and sea makes with the normal are α and γ. These ray which hit the edge of the window determine the size of the illuminated area at the bottom of the sea. X is equal to

$$X = L + 2(H + d)\tan\gamma$$

From Snell's law

$$\sin\gamma = \frac{\sin\alpha}{n_w}$$

$$\Rightarrow \tan\gamma = \frac{\sin\alpha}{\sqrt{n_w^2 - \sin^2\alpha}}$$

where n_w is the index of refraction of the water. From the geometry of the system we have

$$\sin\alpha = \frac{L}{\sqrt{L^2 + 4h^2}}$$

$$\Rightarrow \tan\gamma = \frac{L}{\sqrt{(n_w^2 - 1)L^2 + 4n_w^2 h^2}}$$

Thus,

$$X = L\left(1 + \frac{2(H+d)}{\sqrt{(n_w^2 - 1)L^2 + 4n_w^2 h^2}}\right)$$

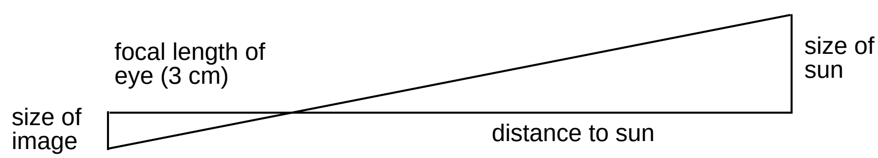
If you were in the sub you would see the illuminated area formed by the continuation of the ray, which is in the sub. Therefore, the length of any side of the square is

$$X' = 2(H+d+h)\tan\alpha = \frac{L(H+d+h)}{h} \qquad \textit{[Yaser]}$$

2) "Out, out damned spot!"

Most of the focussing comes from the curved shape of your eyeball at the *cornea*, and the lens of the eye mostly 'touches up' the focussing. Makes little difference for this question.

a) The focal plane of the eye is located approximately 22 mm behind the cornea or front of the eye. The iris controls the size of the pupil and on a bright day a typical pupil size is 2 mm in diameter. Approximating the eye lens as a "thin lens" we can use Newton's form of the thin lens equation (a similar-triangles law, actually):



$$H_i \,/\, H_o = f \,/\, x$$

Where:
- H_i — height of the image;
- H_o — height of the object (radius of Sun $\cong 7 \times 10^8$ m)
- f — focal length of the lens
- x — distance of the object (distance Sun-Earth $\cong 1.5 \times 10^{11}$ m)

The size of the image on the focal plane of the eye is;

$$H_i = (H_o \bullet f) \,/\, x \cong 10^{-4}\ \text{m}$$

The area of the image on the focal plane is therefore;

$$A \cong \pi \bullet 10^{-8}\ \text{m}^2$$

The pupil of the eye, approximately 2 mm in diameter under bright conditions will allow into the eye approximately;

$$P \cong (3 \bullet 10^3\ \text{W/m}^2) \bullet (\pi \bullet 10^{-8}\ \text{m}^2) \cong 3 \bullet 10^{-3}\text{W}$$

This is about the power of a high-school HeNe laser. The resulting intensity in the focus on the retina of the eye is:

$$I \cong P / A \cong 3 \bullet 10^5 \, W/m^2$$

About 300 times concentrated from direct sunlight. If your pupil has not closed down fast enough, this can be much higher — more like 1000 kW m^{-2}.

b) In the problem above, the image size was determined by geometry — actually by the angle the sun subtends at your eye, which is the angular size, responsible for *parallax*. When you consider a laser with practically parallel light rays, the focus should be essentially a *point*, except for the fact that there is *diffraction* of the light, which makes the final focal spot slightly blurry.

The formula for this is:

$$d = 2 \lambda f / a$$

where d is the focal spot diameter, λ is the wavelength, f is the focal length, and a is the beam diameter at the lens. Often the quantity (f/a) is called the *f-number*. For our case, $\lambda \sim 700$ nm (red light), f = 3 cm, a = 1 mm; then d = 42 μm. The laser power of 5 mW all goes into this spot, for an average intensity of 3.6×10^6 W m^{-2}.

[EXTRAS: How do you understand this formula for diffraction? Only an infinite plane wave propagates without any change at all — but it extends infinitely! For any real wave, there is a general relationship:

$$\Delta x \, \Delta k \sim 1$$

where Δx is the *uncertainty in the position*, and Δk is the *uncertainty in the wavevector*, related to the momentum; it's direction is the direction of propagation. For a plane wave, Δx is infinite, so Δk can be zero — you know exactly where the wave is going (but not at all any special place where it is). When a plane wave passes through a slit, or the pupil of your eye, the hole size sets a limit on Δx, and so Δk cannot be zero. The wave doesn't lose any momentum, going through the hole, so what happens is the *direction* of k spreads out — the wave propagates not just in a straight line, but it diffracts from the aperture. The same thing applies for the focal spot, if you trace backwards: light from a tiny focal spot spreads out faster than from a big focal spot. So tiny focal spots go together with smaller f-numbers.

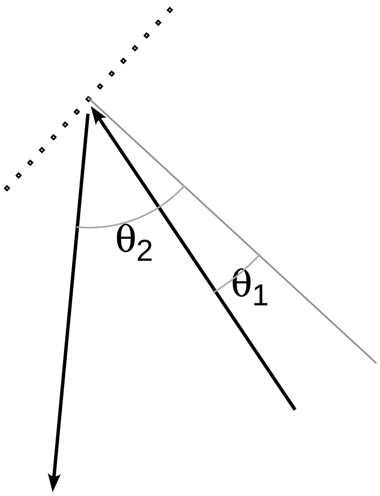
$$\tan\theta = (a/2)/f$$

where θ is the half-angle of the spreading out of the light. Then the formula for d can be written as

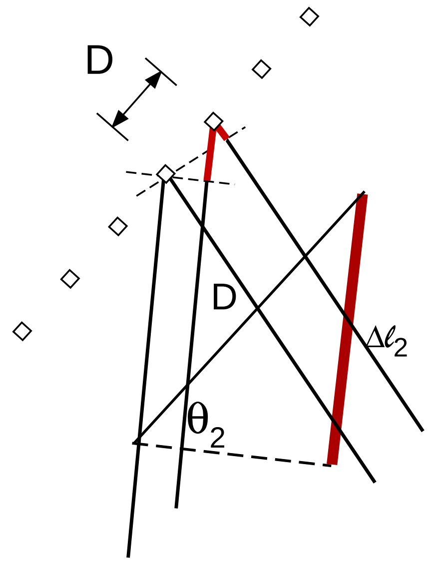
$$d \bullet \tan\theta \sim 1$$

It makes sense, when you realize that $d = \Delta x$ (the width) and $\tan\theta = \Delta k$ (the spreading).]
[Sal & Robin]

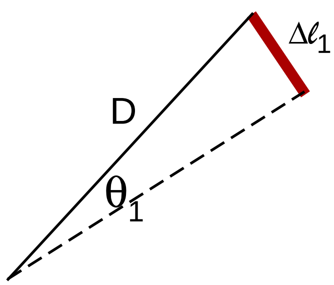
3) Musical buildings



a) The question is pretty easy if you let the distances be large. Then the rays are parallel, and the solution is much like the way you solve for interference in a thin layer, like oil on water — you need to examine the path-differences.



The figure at left shows the basic setup of angles for incoming and outgoing sound waves; I draw them as rays here. The figure at right illustrates the pathlength differences: the straight lines across the path mark where paths are equal, and the extra distance one ray must travel is marked by the fat line (in red, if you have a colour version of this). This extra pathlength is from two parts: extra distance coming in, and extra distance coming out from the building-grating.



The figure at left shows the details for the incoming, and the figure at right shows the detail for the outgoing distance. The total extra path-length is $\Delta\ell_1 + \Delta\ell_2$:

$$\frac{\Delta\ell_1}{D} = \sin\theta_1 \quad \textit{and} \quad \frac{\Delta\ell_2}{D} = \sin\theta_2$$

$$\Delta\ell = \Delta\ell_1 + \Delta\ell_2 = D\cdot(\sin\theta_1 + \sin\theta_2)$$

At the speed of sound C_s, one reflection will be later than the other by a time T:

$$T = \frac{\Delta\ell}{C_s} = \frac{D}{C_s}\cdot(\sin\theta_1 + \sin\theta_2)$$

In fact, all reflections are separated in time by this — the 'bang' of the fireworks burst comes to the listener as a series of pulses, and T is the *period* of this repetitive signal. The frequency is then:

$$\nu = \frac{1}{T} = \frac{C_s}{D\cdot(\sin\theta_1 + \sin\theta_2)}$$

For a building with ribs about 2 m apart (roughly the size of a window), with $C_s = 300$ m s^{-1}, and $\theta_1 = 22°$ and $\theta_2 = 45°$, this gives $\nu = 140$ Hz. That's a fairly low note, but not as low as the famous 40 Hz organ note in Bach's Tocatta and Fugue in D-minor which isn't actually present on many recordings…

It isn't necessary to assume that the distances L_1 and L_2 are very large. If they aren't, and interesting thing happens: not all the periods T between notes are exactly the same, but they increase or decrease as the 'bang' ripples off all the ribs of the building. (This is not difficult to show — just re-draw the diagrams above, with rays coming from a point.) As a result, the note that you hear actually shifts a little while you're listening. The technical term for this is 'frequency chirp', like the whistle of a bird may rise or fall.

b) The fireworks 'bang' has many frequencies in it, so another way to imagine this same problem is to compare it to white light hitting a diffraction grating and spreading out — like a room light diffracting from a compact disc. In that case, it's easy to find what frequency you'll hear, in a given place. It's just the same wavelength formula as for a diffraction grating:

$$n\lambda = D \cdot (\sin\theta_1 + \sin\theta_2)$$

where n is the *order* of diffraction (check this out on a CD, and you'll see several rainbows in the colours from a point-source light).

In terms of sound, the note you hear is usually determined by the lowest frequency, or longest wavelenth — the *fundamental*. The other wavelengths are shorter; these higher frequencies are then multiples of the fundamental, called the *overtones*. These overtones determine the *timbre* or basic sound-quality of an instrument, like flute vs. oboe.

For the fireworks, the fundamental frequency is:

$$\nu = \frac{C_s}{\lambda} = \frac{C_s}{D \cdot (\sin\theta_1 + \sin\theta_2)}$$

which is the same formula we found before. *[Robin]*

4) A high-tech diet, high in optical fiber...

a) There are many paths light can take down the fiber. That ray of light which moves in a straight line parallel to the axis of the fiber reaches the other end of the fiber earlier that some ray which has an angle θ with respect to the axis, and which travels more in a zig-zag pattern to get to the end of the fiber. The time it takes for the straight ray to travel a distance L along the fiber is

$$t_1 = \frac{nL}{c}$$

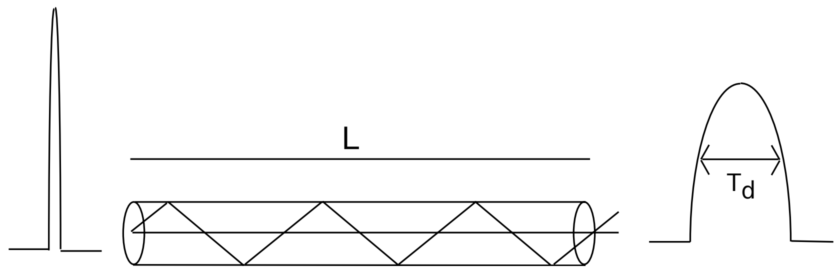
where *c* is the speed of light. The other ray has to travel a longer distance:

$$L/\cos\theta$$

and it gets to the other end at a time

$$t_2 = \frac{nL}{c\cos\theta}$$

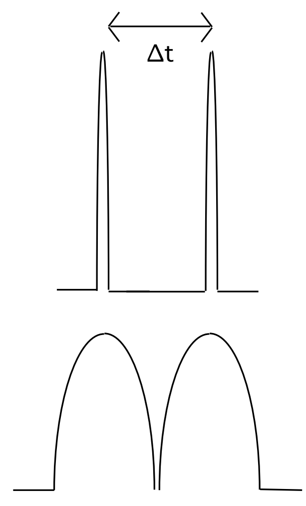
The duration of the pulse at the output of the fiber is usually measured as the width of the pulse measured halfway down from the peak of the pulse — the full-width at half-max (FWHM). To find this exactly, one needs to know how the input light goes into the different rays, but it will be roughly half of the time delay between the first rays to arrive and the last rays to arrive:



$$\tau_d = \frac{t_2 - t_1}{2} = \frac{nL}{2c}\left(\frac{1}{\cos\theta} - 1\right)$$

where τ_d is the pulse duration. This effect is usually called *modal dispersion*. We assumed in our calculation that the initial duration of the input pulse is much smaller that the output pulse, *i.e.*, much smaller than this spreading–out by different rays. Otherwise the time-stretching adds to the original pulse duration the way that two sides of a right-angle triangle give the hypotenuse:

$$\tau = \sqrt{\tau_p{}^2 + \tau_d{}^2}$$



b) Here we again assume that the initial pulse duration is much smaller than the dispersion broadening. In order that we can tell the pulses apart, after they broaden, they shouldn't overlap with each other much. Therefore, the duration of the output pulses should be smaller than roughly half of the time separation between them.

$$\tau_d \le \Delta t/2 \quad \Rightarrow \quad L \le \frac{c\Delta t}{n(1/\cos\theta - 1)}$$

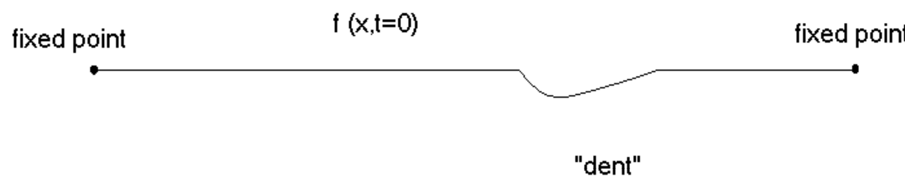
c) The energy of a pulse is equal to the integral of the intensity in terms of the time. If the pulse looks similar to a Gaussian pulse, we can approximate the energy of the pulse by the product of the peak intensity and the pulse duration. If the fiber is lossless, we expect the energy of the output pulse to be equal to the input pulse. Therefore,

$$I_{peak}^{in}\tau = I_{peak}^{out}\tau_d \Rightarrow I_{peak}^{out} = \frac{I\tau}{\Delta t}$$

where $I_{peak}^{in} = I$ and I_{peak}^{out} is the peak intensity of the input and output pulses. We assume that the length of the fiber is the maximum length such that the output pulses are distinguishable. *[Yaser]*

5) Wave goodbye

a) Let us assume that the shape of the "dent" after the stick strikes the clothesline is given by f(x, t=0).



An example of such a "dent," frozen in time, is shown in the figure above.

From the properties of waves given in the question, such a f(x, t=0) can be the instantaneous result of the addition of *two* identically shaped pulses, each *half* the size of the original "dent" — one pulse travelling to the left and the other to the right with the same speed.

$$f(x, t=0) = 1/2\, f_1(x) + 1/2\, f_2(x)$$

where $f_1(x, t=0) = f(x, t=0)$ and $f_1(x, t) = f_1(x+vt)$

$f_2(x, t=0) = f(x, t=0)$ and $f_2(x, t) = f_2(x-vt)$

Once a solution, always a solution: the dent created instantaneously immediately decomposes into these two travelling waves after the line is struck. This explains the two waves you will see emerge, after impact, travelling in opposite directions.

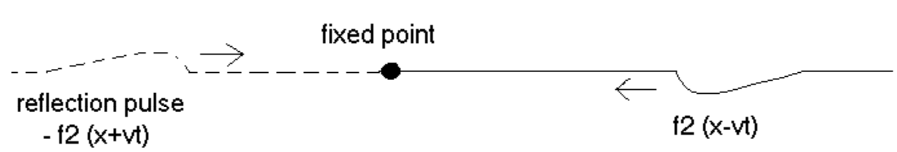
b) The fixed ends of the clothesline impose some conditions on the evolution of the waves. For all times t, the actual wave functions f_1, f_2 must be constant at the fixed ends.

$$f_1(x = L_1, t) = 0 \quad [1]$$

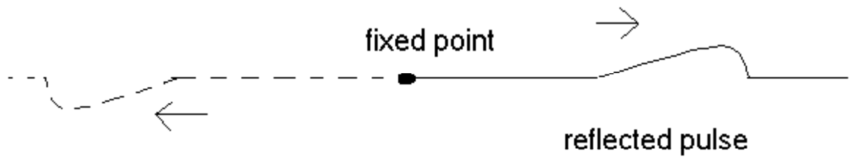
$$f_2(x = -L_2, t) = 0 \quad [2]$$

where L_1 and $-L_2$ are the positions of the fixed ends relative to the position of impact. The following trick can satisfy conditions (1) and (2) while revealing something familiar from experience.

For any wave f_2, imagine a matching wave which is a negative version, and travelling in

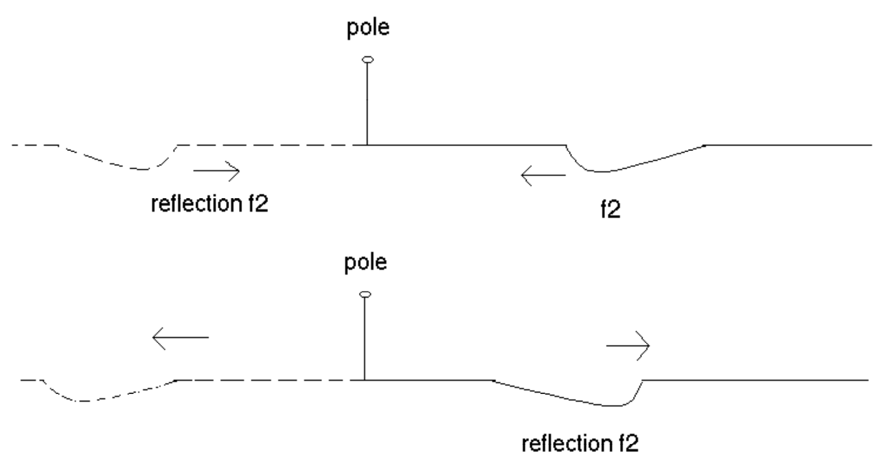


the opposite direction toward the fixed end in some imaginary extension of the wire (see the figure below). There is no fixed point in this picture. As the wave and its negative counterpart approach the spot which *ought* to be fixed, the waves will always add up to zero — satisfying the condition for the fixed point!



What you observe *mathematically* is the two waves passing each other and leaving the tied-point fixed. Since the point in the middle never moved, we could have nailed it down — it's our fixed end. So when a *real* wave goes into a fixed end, the resulting wave is the negative wave coming back from the fixed point, just the solution you'd have from the doppleganger.

c) Using a similar trick as in part (b), we consider superimposing a waves f_2 with a positive clone coming the other way. This will 'explain' or model for us reflections at a sliding ring on a pole. No longer do we have restrictions (1) and (2): at the sliding ring, they're replaced by the condition that there can be *no vertical forces* — only horizontal ones (or the massless ring would immediately move in response and the vertical force would vanish).

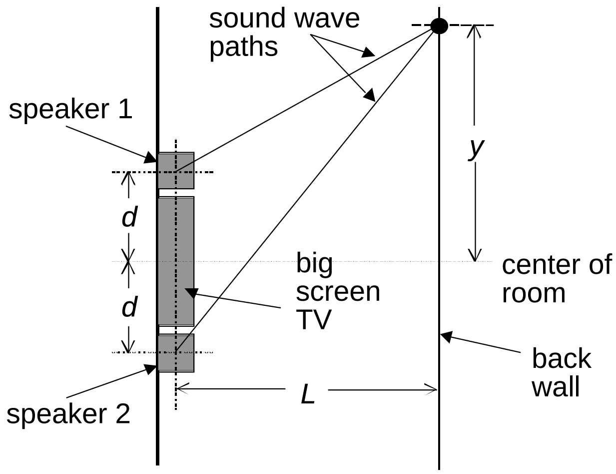


With only a *horizontal* force possible, the tension in the string, for the right solution, must give a horizontal force: so the string itself must be horizontal. That's fine — our doppleganger wave now is always a mirror image of the real one, and therefore *symmetric*. When these two add, the tangent to the string at the place of the pole is always horizontal; this is true for any smooth wave adding with its mirror image.

The waves, as they add, rise to a height of twice the amplitude of each. So the ring at the pole rises to twice the height of the wave that travels to it, and then the wave coming back will be identical to the one going in. *[Sal]*

6) Ten yards for interference…

The Physics: In this question, we do not assume the spacing *2d* between the speakers is very small when compared to the distance *L* from the speakers to the back wall. If the difference in path length from speaker 1 to point *P* and speaker 2 to point *P* is an integral multiple of the sound wavelength, there will be constructive interference.



a) Path lengths from speaker 1 to point *P* and speaker 2 to point *P* can be determined using the Pythagorean theorem:

Speaker 1 to point *P*: $d_1 = \sqrt{(y-d)^2 + L^2}$

Speaker 2 to point *P*: $d_2 = \sqrt{(y+d)^2 + L^2}$

Therefore, the difference in path length is:

$y \geq 0$: $\delta = d_2 - d_1 = \sqrt{(y+d)^2 + L^2} - \sqrt{(y-d)^2 + L^2}$

$y \leq 0$: $\delta = d_1 - d_2 = \sqrt{(y-d)^2 + L^2} - \sqrt{(y+d)^2 + L^2}$

For constructive interference, the difference in path length must be an integral multiple of the wavelength ($\delta = m\lambda$):

$$y \geq 0: \quad \sqrt{(y+d)^2 + L^2} - \sqrt{(y-d)^2 + L^2} = m\lambda = m\frac{v}{f} \qquad (1)$$

$$y \leq 0: \quad \sqrt{(y-d)^2 + L^2} - \sqrt{(y+d)^2 + L^2} = m\lambda = m\frac{v}{f} \qquad (2)$$

where $m = (0,1,2,\ldots,\infty)$

In equations (1) and (2), *m* is the order of the interference (*i.e.*, n=2 means they interfere by being *two* waves out of phase, *f* is the frequency of the sound wave, *v* is the speed of sound the medium and λ is the wavelength of sound in the medium.

b) We need to solve for *y* using equations (1) and (2) when L = 5 m, d = 3 m, f = 320 Hz, v = 331 m/s:

$$y \geq 0: \quad \sqrt{(y+3)^2 + 5^2} - \sqrt{(y-3)^2 + 5^2} = m\frac{331}{320}$$

$$y \leq 0: \quad \sqrt{(y-3)^2 + 5^2} - \sqrt{(y+3)^2 + 5^2} = m\frac{331}{320}$$

The first maximum occurs when $m = 0$:

$$y \geq 0: \qquad \sqrt{(y+3)^2 + 5^2} - \sqrt{(y-3)^2 + 5^2} = (0)\frac{331}{320} = 0$$

$$y \leq 0: \qquad \sqrt{(y-3)^2 + 5^2} - \sqrt{(y+3)^2 + 5^2} = (0)\frac{331}{320} = 0$$

by inspection, both equation yield $y = 0$ at the first maximum. The second and third maximum occur when $m = 1$:

$$y \geq 0: \qquad \sqrt{(y+3)^2 + 5^2} - \sqrt{(y-3)^2 + 5^2} = (1)\frac{331}{320} = \frac{331}{320} \tag{3}$$

$$y \leq 0: \qquad \sqrt{(y-3)^2 + 5^2} - \sqrt{(y+3)^2 + 5^2} = (1)\frac{331}{320} = \frac{331}{320} \tag{4}$$

Not everyone will find it easy to determine a regular *closed form* solution for y in equation (3) and equation (4) (some did!). There are, however, a number of easy ways to solve these equations:

- plug in values for y in the equation and iteratively converge to zero in on a solution
- graph the left hand side and right hand side the equation to see where the curves intersect
- use a math program to solve the equation numerically.

I have solved equation (3) and equation (4) numerically using a math program called Maple. The answers I got are:

$y \geq 0$: $\qquad y = 1.02$ m

$y \leq 0$: $\qquad y = -1.02$ m

Therefore, the three points of maximum intensity closest to the centerline are:

y = {–1.02 m, 0 m, 1.02 m} *[Brian]*