

2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 1: General

1) Let the sun shine in!

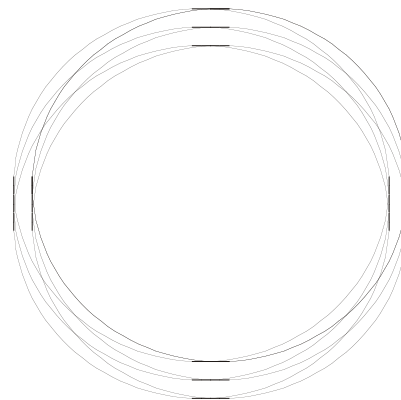
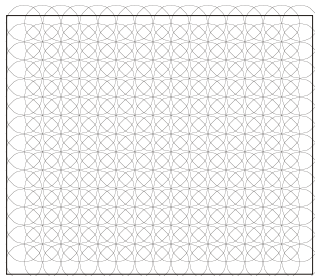
Let's say the mirror is $0.5 \times 0.5 \text{ cm}^2$ square. It is important that your mirror shouldn't be round to observe the effect. When the wall is close to the mirror the bright spot has the same shape as the mirror, in this case square. When you go farther and farther it changes gradually to the circle. This effect comes from the fact that the rays coming from the sun are not exactly parallel due to the size of the sun.

We can solve this question by considering that the square shape mirror consists of infinitely large number of very tiny mirrors. Each tiny mirror makes a circular image of the sun on the wall. If the distance between wall and the mirror is small all these small circular images will form together the shape of the mirror. At large distances, these circular images are big and they all overlap with each other form a big circle. By geometry, the ratio between the radius of the image to the distance between the wall and the mirror is equal to the ratio between the radius of the sun and the distance of sun to earth. For $0.5 \times 0.5 \text{ cm}^2$ mirror, up to about 50 cm the spot on the wall looks square. As we go farther the spot changes to circle and at about 1.5 meter away, the circle has the radius of about 0.75 cm. Therefore the ratio between diameter of the sun and its distance to the earth is about 0.01.

The distance x where the image shape starts changing from square to the circle, is about where you get the smallest image and by geometry

$$\frac{x}{\text{Diameter of the mirror}} = \frac{\text{Distance between sun and earth}}{\text{Diameter of the sun}}$$

which is about half a meter in this case.



(Right) At small distance between the mirror and the wall in which the circles form the square .

(Left) At large distance between the mirror and the wall in which the circles overlap and the whole image looks like a circle.

2) Ping-Pong!

(a) A table tennis paddle is much heavier than the ping-pong ball. Therefore we can assume that its mass is infinity compared to the ball. Therefore, in the wall's frame (or that of a table tennis paddle!), the velocity of the ball is the same before and after it hits the wall, but in the opposite direction. The velocity of the ball in this frame is $v+u$ and so it bounces back with $v+u$ relative to the wall. Therefore its velocity relative to the ground is $v+2u$.

(b) The change in the kinetic energy is

$$\Delta K = K_f - K_i = \frac{1}{2}m(v + 2u)^2 - \frac{1}{2}mv^2 = 2mu(u + v)$$

where m is mass of the ball and K is the kinetic energy.

(c) The change in the linear momentum of the ball is

$$\Delta P = P_f - P_i = m(v + 2u) - (-mv) = 2m(v + u)$$

Therefore if the collision time is ΔT , then the force is

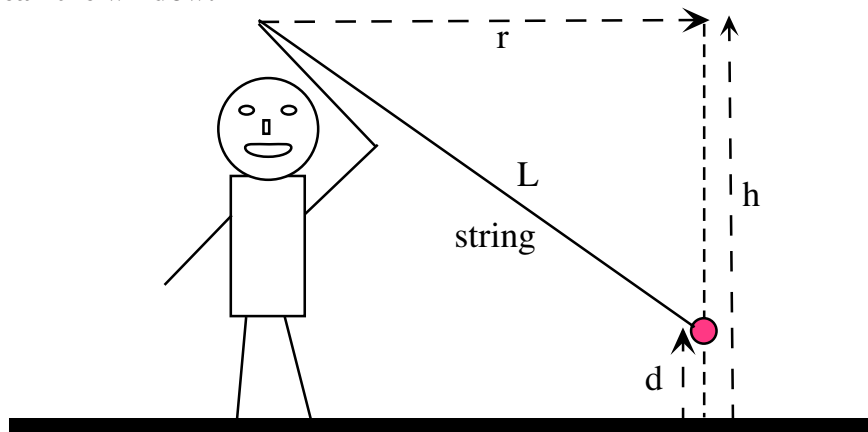
$$F = \frac{\Delta P}{\Delta T} = \frac{2m(u + v)}{\Delta T}$$

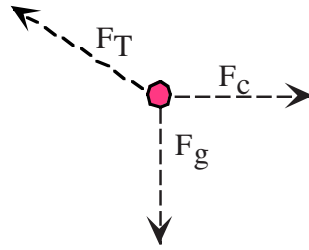
The distance that the ball moves during the collision is equal to the distance that the wall moves which is $u\Delta T$. Therefore the work done by the wall on the ball is

$$W = Fu\Delta T = 2mu(u + v)$$

which is equal to the change in the kinetic energy of the ball as we expect.

3) Don't break the window!





The faster you spin the ball around you, the higher it will go, but the more stress we put on the string. Looking at the above diagrams, we see that

$$\frac{h-d}{r} = \frac{F_g}{F_c} = \frac{mg}{m\omega^2 r} \Rightarrow d = h - \frac{g}{\omega^2}$$

This is independent of the length of the string, and as we intuitively feel, $d \rightarrow h$ as $\omega \rightarrow \infty$. The tension in the string is, however, proportional to the length of the string:

$$\frac{L}{h-d} = \frac{F_T}{F_g} = \frac{F_T}{mg} \Rightarrow F_T = \frac{mgL}{h-d} = m\omega^2 L$$

(a) Rearranging the above equation, the length of the string

$$L = \frac{F_T(h-d)}{mg}$$

So L is a maximum when F_T is a maximum and d is a minimum, *i.e.* when $F_T=T$ and $d=0$.

$$\therefore L_{\max} = \frac{Th}{mg}$$

(b) The distance the ball flies is equal to its horizontal velocity, $v=r\omega$, multiplied by the time it takes to fall to the ground, t . If it starts to fall at a height, d , then the time it takes to hit the ground is

$$d = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2d}{g}}$$

What we need to do is express the distance of travel, $s=vt$, in terms of L , and find the value of L which maximizes s .

$$s = vt = r\omega \sqrt{\frac{2d}{g}} = \sqrt{r^2 \omega^2 \frac{2d}{g}}$$

The maximum distance will occur for the maximum tension, $F_T=T$, and by geometry

$$r^2 = L^2 - (h-d)^2$$

and using previous results

$$\omega^2 = \frac{F_T}{mL} = \frac{T}{mL} \quad \text{and} \quad d = h - \frac{g}{\omega^2} = h - \frac{g}{\left(\frac{F_T}{mL}\right)} = h - \frac{mgL}{T}$$

we have

$$s = \sqrt{\left(L^2 - \left(h - \left(h - \frac{mgL}{T} \right) \right)^2 \right) \left(\frac{T}{mL} \right) \frac{2(h - mgL/T)}{g}}$$

$$= \sqrt{2L \left(1 - \left(\frac{mg}{T} \right)^2 \right) \left(\frac{hT}{mg} - L \right)}$$

Looking at this we easily see that the maximum distance will be when we maximize

$$L \left(\frac{hT}{mg} - L \right)$$

This is just a parabola in L with an obvious maximum at

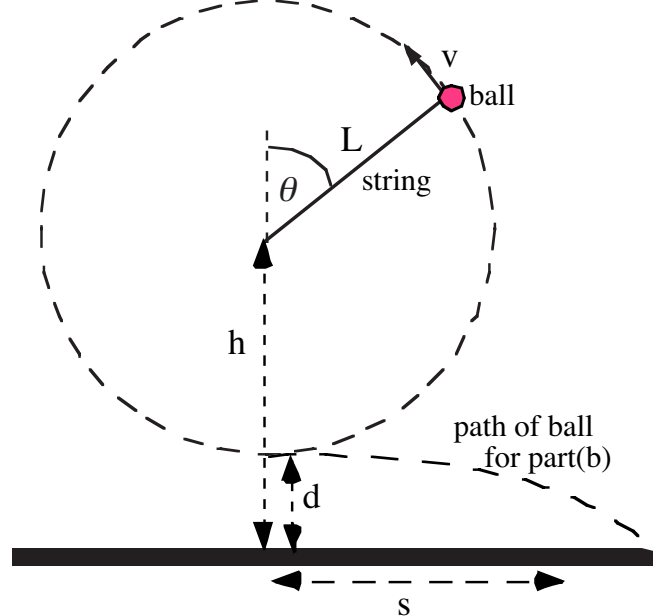
$$L_{\max} = \frac{hT}{2mg}$$

For our interest, we can now calculate the maximum distance

$$s_{\max} = \frac{h}{\sqrt{2}} \sqrt{\frac{T^2}{m^2 g^2} - 1}$$

Note

Some people assumed the motion of the ball was vertical instead of horizontal. You can't actually "swing the ball in circles around you" in a vertical plane, but semantics aside, let's see what happens if you swing the ball in a vertical plane beside you.



- The ball must orbit fast enough so the centrifugal force can overcome gravity at the top of its orbit, so the velocity of the ball at the top of the orbit must be such that $v^2/L > g$.
- If the string breaks, it will break at the bottom of the orbit since the tension must be a maximum since gravity and the centrifugal force add together at that point.

To figure this out in detail, let's assume you have got the ball going and it is now orbiting with no further energy input from you. The constant total energy of the ball is

$$\begin{aligned}
 E_{tot} &= E_{potential} + E_{kinetic} \\
 &= mgL\cos\theta + \frac{1}{2}mv^2
 \end{aligned}$$

where we have set the gravitational potential energy zero at the height h . The velocity of the ball thus depends on where in the orbit it is

$$v^2 = 2 \frac{E_{tot} - mgL\cos\theta}{m}$$

Not surprisingly, the maximum velocity is at the bottom of the orbit ($\theta=180^\circ$).

The tension in the string must balance the sum of the centrifugal force plus the component of the gravitational force along the string:

$$\begin{aligned}
 F_T &= m \frac{v^2}{L} - mg\cos\theta \\
 &= \frac{2(E_{tot} - mgL\cos\theta)}{L} - mg\cos\theta \\
 &= \frac{2E_{tot}}{L} - 3mg\cos\theta
 \end{aligned}$$

As expected, this shows the tension is a maximum at the bottom of the orbit and a minimum at the top; to minimize the tension we want E_{tot} to also be a minimum. The tension at the top ($\theta=0^\circ$) cannot be less than zero or the string will not be taut and the ball will fall. So we want

$$\begin{aligned}
 E_{tot} &= \frac{3}{2}mgL \\
 \therefore F_T &= 3mg - 3mg\cos\theta = 3mg(1 - \cos\theta)
 \end{aligned}$$

The tension at the bottom must be less than the breaking tension, T , so we must have

$$T > 6mg$$

This is independent of the length of the string, so as long as the string is strong enough to support $6mg$, the maximum length of the string is just set by the height you hold the end of the string above the ground, *i.e.*

$$L_{max} = h$$

If we take the same string and swing it horizontally, we have $L_{max} = (T/mg)h = (6mg/mg)h = 6h$, so you can swing a much longer string horizontally instead of vertically. Even the weakest string that can support the weight of the ball, *i.e.* $T > mg$, has $L_{max} > h$ if swung horizontally.

Finally, let's check how far the ball will fly if the string breaks when swinging vertically. Since the tension is maximum at the bottom, the ball will fly horizontally from the bottom of the orbit and the time to hit the ground is

$$t = \sqrt{\frac{2d}{g}} = \sqrt{\frac{2(h-L)}{g}}$$

Again we want to maximize $s=vt$,

$$\begin{aligned}
s &= vt \\
&= \sqrt{2 \frac{E_{tot} - mgL \cos(180^\circ)}{m}} \sqrt{\frac{2(h-L)}{g}} \\
&= \sqrt{4 \frac{\frac{3}{2}mgL + mgL}{m} \left(\frac{h-L}{g}\right)} \\
&= \sqrt{10L(h-L)}
\end{aligned}$$

So the maximum value is $L=h/2$, and the maximum distance the ball will fly in the vertical orbit case is

$$s_{\max} = \sqrt{\frac{5}{2}}h$$

Comparing this with the horizontal orbit, we see that the minimum strength string strong enough to be swung¹ vertically (*i.e.* $T > 6mg$) will fly exactly the same distance if swung horizontally or vertically. In the vertical case, however, the maximum distance is independent of the string strength, but in the horizontal case the maximum distance increases with the strength of the string without limit (except for mundane factors such as air resistance and the size of the earth).

4) Cowabunga!

- (a) For this question, it is convenient to assume that the cow is a sphere, since that is a shape we know how to deal with and we only want an estimate. The capacitance of an conducting object is the ratio of its electric charge to its electric potential or “voltage”, *i.e.*

$$C = \frac{Q}{\phi}$$

By Gauss’s Law and symmetry, the potential at the surface of a conducting sphere with charge Q and radius r is the same as the potential a distance r from of a point charge Q :

$$\phi = \frac{q}{4\pi\epsilon_0 r}$$

(Where, by convention, $V=0$ at infinity defines the zero potential.) The capacitance of a sphere of radius r is therefore:

$$C = \frac{Q}{\phi} = 4\pi\epsilon_0 r$$

The radius of a sphere with the same volume, V , as a Holstein² cow (mass $M=700\text{kg}$, density $\rho=1000\text{kg/m}^3$) is

$$r = \left(\frac{V}{\frac{4}{3}\pi}\right)^{1/3} = \left(\frac{M/\rho}{\frac{4}{3}\pi}\right)^{1/3} = \left(\frac{700\text{kg}/1000\text{kg/m}^3}{\frac{4}{3}\pi}\right)^{1/3} = 0.55\text{m}$$

The cowcapitance is thus

¹ Try saying “the minimum strength string strong enough to be swung” 5 times fast.

² <http://www.holstein.ca/>

$$C = 4\pi\epsilon_0 r = 4\pi(8.854187817 \times 10^{-12} \text{ F/m})0.55 \text{ m} = 60 \text{ pF}$$

(While writing this problem, I discovered that the standard capacitance of an isolated cow – I am not making this up – is 0.1nF, in good agreement with our estimate. If the cow is standing in a field, the earth acts as a conducting plane and this increases the capacitance to about 0.2nF, but we don't care about a factor of 2 in this problem.)

Note that you had to provide the value of ϵ_0 ; we do not always give the values of basic constants available in any appropriate physics textbook.

- (b) A sphere is not a useful approximation for resistance, since it is hard to calculate the resistance of a sphere. We'll assume a cube, since the resistance of a cube of side length L and resistivity ρ is very simple:

$$R = \rho \frac{L}{A} = \rho \frac{L}{L^2} = \frac{\rho}{L}$$

For cow-size cube

$$L = \left(\frac{M_{\text{cow}}}{\rho_{\text{cow}}} \right)^{1/3} = 0.88 \text{ m}$$

The capacitance of a cow just depends on the cow being a conductor of a certain shape, but resistivity depends on the internal structure of a cow. My rough estimate of the average resistivity is based on the values given for people's insides in the POPBits™ is $\rho_{\text{cow}} \sim 5\Omega \cdot \text{m}$. The resistance is dominated by the low resistance pathways (e.g. the blood vessels), so the high resistivity of bones and fat are largely irrelevant. So my estimate of the resistance of the cow is just

$$R = \frac{5\Omega/\text{m}}{0.88\text{m}} = 6\Omega \sim 10\Omega$$

This is much cruder than our capacitance estimate. This is because parallel capacitances add, so the size of the object pretty much determines the total capacitance; parallel resistances add inversely, so any small scale structure (e.g. blood vessels, noses) can have a big effect. The resistance is also sensitive to how the cow is connected in a circuit. In this case the resistance of the cow is probably dominated by the cow's nose and is probably an order of magnitude higher than our simple estimate since the cow's nose is an order of magnitude smaller than the whole cow. (Note: This seems consistent with the 150 Ω I get when I hold one probe in each hand and measure my resistance with a multimeter, I have not yet tried to measure the resistance of my nose.)

- (c) The peak power is just the initial power, since the cow's voltage drops as it discharges:

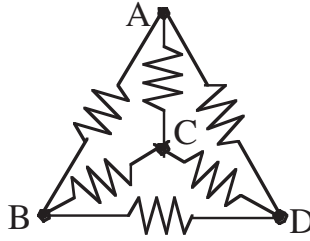
$$P_{\text{peak}} = I_{\text{peak}} V_{\text{peak}} = \frac{V_{\text{peak}}^2}{R} = \frac{V_{\text{initial}}^2}{R} = \frac{(10\text{kV})^2}{10\Omega} \sim 10 \text{ MW}$$

No wonder static hurts!

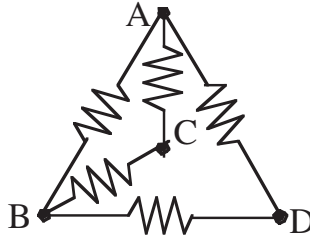
Note: By mistake, a draft pdf version of this problem set was originally posted with 600kg cows at 20KV, instead of 700kg cows at 10KV as in the print version. This is an approximate problem so either set of numbers are equally good.

5) Symmetry Cubed!

- (a) By symmetry, the resistance between any two vertices must be the same, so let's choose vertices to measure the resistance between A and B.



By symmetry, C and D will have the same potential so no current will flow between them, so that resistor can be ignored and the tetrahedron is equivalent to

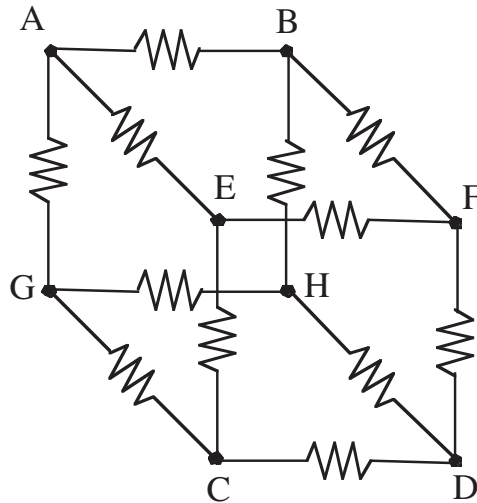


So the total resistance between A and B can be calculated by adding the 3 paths in parallel, i.e.

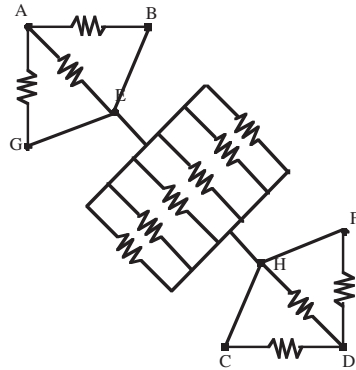
$$\frac{1}{R_{AB}} = \frac{1}{R} + \frac{1}{2R} + \frac{1}{R} = \frac{2}{R}$$

$$\therefore R_{AB} = \frac{R}{2}$$

(b) First we calculate the resistance between A and D.



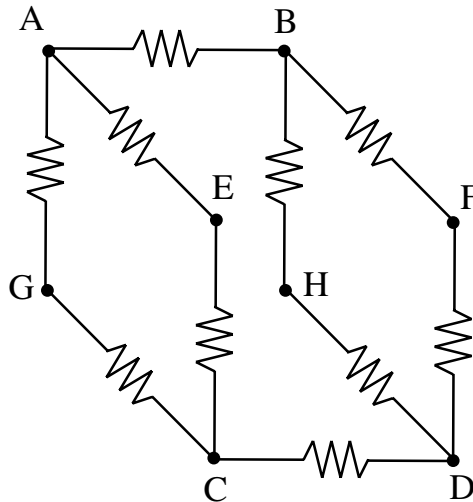
There is lots of symmetry, so it is easy to see that G, E, & B will have the same potential and can be treated as the same point, and similarly C, H, & F are at the same potential and can be treated as the same point, so the circuit is equivalent to



So we see that

$$R_{AD} = \frac{R}{3} + \frac{R}{6} + \frac{R}{3} = \frac{5R}{6}$$

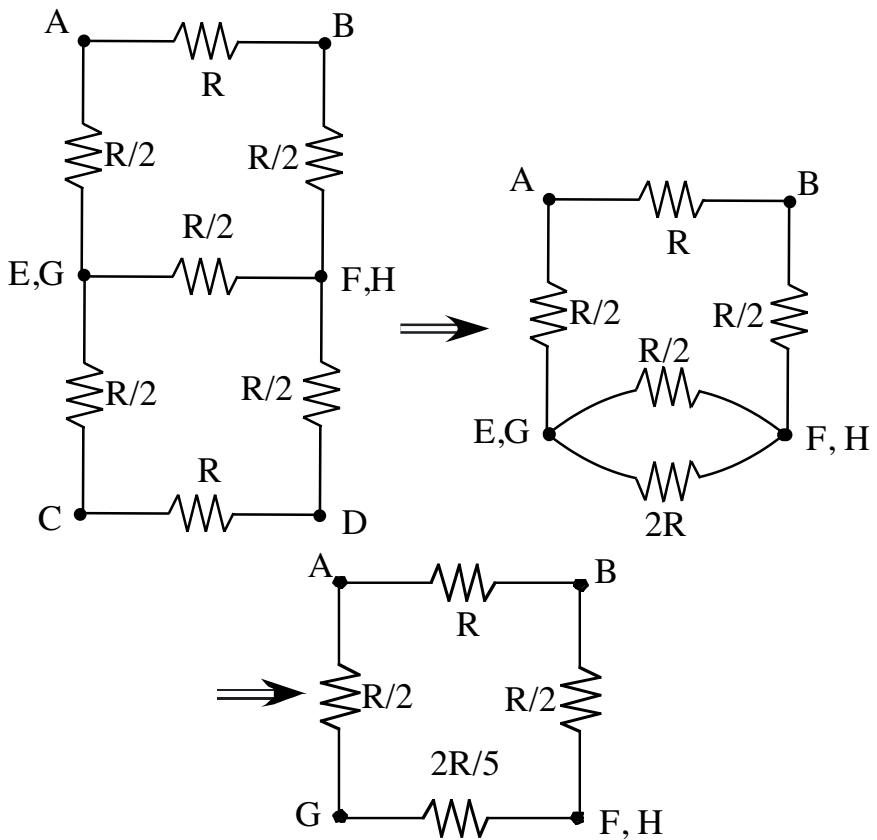
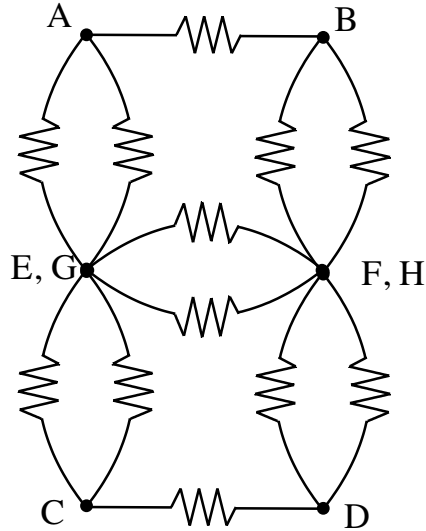
(c) Now we want the resistance between A and C. By symmetry, E, F, G, & H are all at the same potential. As with part (a), the circuit is not affected by ignoring the resistors connecting points at the same potential, so our equivalent circuit is:

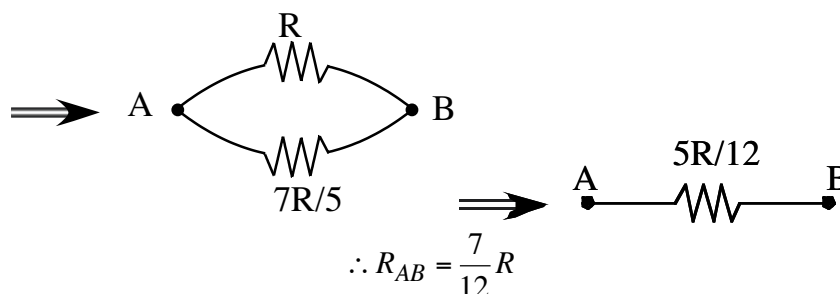


So

$$\begin{aligned} \frac{1}{R_{AC}} &= \frac{1}{R_{AEC}} + \frac{1}{R_{AGC}} + \frac{1}{\left(R_{AB} + \left(\frac{1}{R_{BFD}} + \frac{1}{R_{BHD}} \right)^{-1} + R_{CD} \right)} \\ &= \frac{1}{2R} + \frac{1}{2R} + \frac{1}{\left(R + \left(\frac{1}{2R} + \frac{1}{2R} \right)^{-1} + R \right)} \\ &= \frac{1}{R} + \frac{1}{(R + R + R)} \\ &\therefore R_{AC} = \frac{3}{4}R \end{aligned}$$

(d) Now we want the resistance between A and B. This is the toughest one yet, but there is still some symmetry, and we see that vertices E & G are at the same potential and can be connected without changing the circuit, and F & H are at the same potential and be connected. Our new equivalent circuit is





Note: You can, of course, solve all these systems using Kirchoff's rules and Ohm's law to write down systems of linear equations which can be solved by substitution or matrix algebra. This is fine, but we thought using symmetry was more fun.

6) Is it hot enough for you?

(a) The activity A is

$$A = \frac{N}{\tau_{1/2} / \ln 2}$$

where N is the number of atoms and $\tau_{1/2}$ is the half-life of the radioactive isotope. The number of atoms for each isotope is simply its mass divide by its atomic weight: 15 atomic mass units (amu) for Carbon-14 and 40 amu for Potassium-40. For a 70kg person the masses of each isotope are:

$$m_{C14} \sim 70\text{kg} \times 19\% \times 10^{-12} \sim 1.33 \times 10^{-11}\text{kg} = 13\text{ng}$$

$$m_{K40} \sim 70\text{kg} \times 0.3\% \times 0.0117\% \sim 2.46 \times 10^{-5}\text{kg}$$

Therefore the activities are

$$A_{C14} \sim \frac{1.33 \times 10^{-11}\text{kg}}{(5730\text{yr} / \ln 2)} \frac{1}{(14\text{amu})(1.66 \times 10^{-27}\text{kg} / \text{amu})} = 2193/\text{s}$$

$$A_{K40} \sim \frac{2.46 \times 10^{-5}\text{kg}}{(1.277 \times 10^9\text{yr} / \ln 2)} \frac{1}{(40\text{amu})(1.66 \times 10^{-27}\text{kg} / \text{amu})} = 6363/\text{s}$$

So the total activity is $A_{\text{total}} = 8600$ decays per second.

(b) About half the energy is deposited in the person's body, therefore the power deposited by each isotope is:

$$P_{C14} = \frac{1}{2} A_{C14} (0.16 \times 10^6\text{eV}) (1.60 \times 10^{-19}\text{J} / \text{eV}) = 2.8 \times 10^{-11}\text{W}$$

$$P_{K40} = \frac{1}{2} A_{K40} (1.3 \times 10^6\text{eV}) (1.60 \times 10^{-19}\text{J} / \text{eV}) = 6.63 \times 10^{-10}\text{W}$$

So the total activity is about $P_{\text{total}} = 0.7\text{nW}$.

(c) In thermal equilibrium you will radiate as much heat, P , as you produce, so your temperature, T , will be

$$T = \left(\frac{P}{\epsilon A \sigma} \right)^{1/4}$$

where A is your surface area (about 2m^2), and σ is the Stefan-Boltzmann constant. Your emissivity, ϵ , is less than the black body value of 1, but it doesn't matter too much since the temperature only varies as the 4th root, so we'll assume it is 1. So

$$T \sim \left(\frac{0.7\text{nW}}{2\text{m}^2 (5.670400 \times 10^{-8} \text{W/m}^2/\text{K}^4)} \right)^{1/4} = 0.3\text{K}$$

The cosmic microwave background radiation (<http://background.uchicago.edu/~whu/beginners/introduction.html>) is 2.7K so it will keep you warmer than than your internal radiation.