2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 2: Mechanics

1) Salty log!

If the log has a density greater than the fresher surface water but less than the deeper saltier water, it will float at the halocline such that the net sinking force on the fraction, *f*, of the volume of the log above the halocline will be equal and opposite to the upward buoyancy force on the fraction, 1-*f*, below the halocline, *i.e.*

$$fV(\rho_L - \rho_S) = (1 - f)V(\rho_D - \rho_L)$$
$$\therefore f\rho_L - f\rho_S = \rho_D - \rho_L - f\rho_D + f\rho_L$$
$$\therefore -f\rho_S = \rho_D - \rho_L - f\rho_D$$
$$\therefore f = \frac{\rho_D - \rho_L}{\rho_D - \rho_S}$$

As a quick check of our calculation, we can consider the case before the log begins to sink and it is less dense than the fresh water. In this case it will float on the surface with a fraction in the water

$$f = 1 - \frac{\rho_S - \rho_L}{\rho_S - \rho_{Air}} = \frac{\rho_L}{\rho_S}$$

which is exactly what we expect.

2) Don't leave me!

(a) The ratio of the forces of the earth and the sun on the moon are

$$\frac{F_{Sun}}{F_{Earth}} = \frac{\left. G_N m_{Moon} m_{Sun} \middle/ r_{Sun}^2 \right.}{\left. G_N m_{Moon} m_{Earth} \middle/ r_{Earth}^2 \right.}$$

$$= \frac{\left. m_{Sun} \middle/ r_{Sun}^2 \right.}{\left. m_{Earth} \middle/ r_{Earth}^2 \right.} = \frac{\left. 2.0\times10^{30}\,\mathrm{kg} \middle/ \left(1.5\times10^{11}\,m\right)^2 \right.}{\left. 6.0\times10^{24}\,\mathrm{kg} \middle/ (400000\,km)^2 \right.} = 2.4$$

(b) This question is not trivial, since the three-body problem is not exactly soluble for a inverse-square law force. We cannot write down the exact general equations of motion for an arbitrary system of three (or more) bodies interacting by gravity, so we cannot prove that the earth and the moon will never part company, even if we assume they are ideal point masses. But if they do part company, it certainly won't be for a very, very, very, very long time. This is because the acceleration of the moon due to the sun is the same, on average, as the acceleration of the earth due to the sun, so once they are moving together around the sun (as they do) they will continue to move together. The maximum difference in acceleration between the earth and the moon towards the sun is much smaller than the acceleration of the moon due to the earth.

$$\begin{aligned}\frac{a_{Earth-Sun}-a_{Moon-Sun}}{a_{Moon-Earth}} &= \frac{G_N m_{Sun}\big/ r_{Sun}^2 - G_N m_{Sun}\big/ \left(r_{Sun}-r_{Earth}\right)^2}{G_N m_{Earth}\big/ r_{Earth}^2} \\ &= \frac{m_{Sun}\left(1\big/ r_{Sun}^2 - 1\big/ \left(r_{Sun}-r_{Earth}\right)^2\right)}{m_{Earth}\big/ r_{Earth}^2} \\ &\approx \frac{2m_{Sun}r_{Earth}^3}{m_{Earth}r_{Sun}^3} \\ &= \frac{2\times 2.0\times 10^{30}\,\mathrm{kg}\left(400000km\right)^3}{6.0\times 10^{24}\,\mathrm{kg}\left(1.5\times 10^{11}m\right)^3} = 0.013\end{aligned}$$

For the moon to separate from the earth, it would have to gain or loose angular momentum from the sun or the earth and there is no simple mechanism to do so.

3) Singing in the rain!

(a) We make the approximation that we are vertical rectangular cube with a top surface area *T* and a front (or back) surface area *F*. The time it takes us to reach the shelter is $t=L/v$, so our "top wetness" (*i.e.* the amount of rain landing on our top) is

$$W_T = nTt = nTL/v$$

and the rain we sweep out of the air onto our front is

$$W_F = \rho FL = (n/u)FL$$

where, ρ, is the volume fraction of the air taken up by rain drops.

Our total wetness is

$$W = W_T+W_F = nTL/v + (n/u)FL = nL(T/v + F/u)$$

So our minimum wetness is when we run as fast as we can, *i.e.*

$$v= v_{max}$$

(b) If we have an umbrella, then $W_T=0$ and our total wetness will just be

$$W = W_F = (n/u)FL$$

So wetness is independent of our speed, v.

(c) If we have an umbrella, and the rain is falling at an angle (along our Front/Back plane) then the amount of rain reaching our front or back is not just the amount we sweep out as we walk, but also includes a contribution due to the horizontal movement of the rain,

$$\begin{aligned}W &= W_F = (n/u)FL + (n\tan\theta)Ft = (n/u)FL + (n\tan\theta)FL/v \\ &= nFL\,(1/u + \tan\theta/v)\end{aligned}$$

where $n \tan\theta$ is the rate at which the rain is falling horizontally. If $\theta = 0$ then we just recover our answer to part (b) as expected.

If $\theta > 0$, then the rain is blowing in our face, $(1/u + \tan\theta/v) > 0$, so the best we can do is run as fast as we can, *i.e.* $v = v_{max}$, as for part (a), and our total wetness will be

$$W = nFL\,(1/u + \tan\theta/v_{max})$$

If $\theta < 0$, then the rain is blowing from behind, $(1/u + \tan\theta/v) > 0$, so we can keep dry if we just walk at the horizontal speed of the rain so that it is falling vertically in our reference frame, *i.e.* our optimum speed is $v = -u \tan\theta = |u \tan\theta|$ and our total wetness is W=0.

(Note: We want to get to the shelter so we assume v is positive. If our goal is just not to get wet, an alternate for$\theta > 0$ is just to walk away from the shelter with speed $v = -u \tan\theta = -|u \tan\theta|$)

4) Sproing!

The total force on the block along the ramp is the sum of the spring, gravitational, and frictional forces:

$$F = m\frac{d^2x}{dt^2} = kx + mg \sin\alpha - \mu_k mg \cos\alpha$$

where x is the displacement of the block along the ramp from the position where the spring force is zero, and $\hat{v} = \frac{dx/dt}{|dx/dt|}$ is the unit direction of velocity of the block along the ramp. *i.e.* The frictional force always opposes the direction of motion, and we have made the usual (but not always accurate) textbook assumption that sliding friction is independent of speed.

Without the frictional damping term, it is just an harmonic oscillator

$$m\frac{d^2x}{dt^2} = kx + mg\sin\alpha$$

which you are expected to recognize after some rearrangement

$$\frac{d^2\left(x + \frac{mg\sin\alpha}{k}\right)}{dt^2} = \frac{k}{m}\left(x + \frac{mg\sin\alpha}{k}\right)$$

and know the solution to be

$$\left(x + \frac{mg\sin\alpha}{k}\right) = x_0 \sin(\omega t), \qquad \omega = \sqrt{k/m}$$

$$\therefore\ x = x_0 \sin(\omega t) - \frac{mg\sin\alpha}{k}$$

where x_0 is a (as yet unknown) constant. The block's velocity (in the zero friction limit) is thus

$$\frac{dx}{dt} = \omega x_0 \cos(\omega t)$$

We know the initial velocity at t=0 is v (I probably should have said v_0), so $x_0 = \frac{v}{\omega}$. The minimum (*i.e.* lowest) and maximum positions of the block (in the zero friction limit) are

$$x_{\min} = -x_0 - \frac{mg\sin\alpha}{k}, \qquad x_{\max} = x_0 - \frac{mg\sin\alpha}{k}$$

which will occur at times

$$\omega t_{\min} = \frac{\pi}{2} + 2n\pi, \qquad \omega t_{\min} = \frac{3\pi}{2} + 2n\pi, \qquad n = 0,1,2,\ldots$$

The total kinetic (T) and potential (U) energy of the block are:

$$\mathrm{T} = \frac{1}{2}\mathrm{m}\left(\frac{dx}{dt}\right)^2$$

$$\mathrm{U} = \frac{1}{2}\mathrm{k}x^2 + \mathrm{mg}\ sin\alpha\ x$$

where we have chosen the (arbitrary) zero for the gravitation potential energy to be the initial point x=0 so that the total potential energy is zero at the initial point.. The initial kinetic energy is

$$\mathrm{T}_0 = \frac{1}{2}\mathrm{mv}^2$$

and we can find the maximum extension points where the block's motion reverses itself since the velocity and hence kinetic energy at these points is zero so all the energy must in the form of potential energy, *i.e.*

$$\frac{1}{2}mv^2 = \frac{1}{2}kx_{\min}^2 + mg\sin\alpha\ x_{\min}$$

$$\therefore x_{\min} = \frac{-mg\sin\alpha \pm \sqrt{(mg\sin\alpha)^2 - 4\left(\frac{1}{2}k\right)\left(-\frac{1}{2}mv^2\right)}}{2\left(\frac{1}{2}k\right)}$$

$$= \frac{-mg\sin\alpha \pm \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}$$

By definition, x_{min} must be less than zero, so only one of the two solutions is possible, *i.e.*

$$x_{\min} = \frac{-mg\sin\alpha - \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}$$

(a) The frictional force is small, so the motion of the block is damped only very slowly. The kinetic energy lost is just the integrated frictional energy loss. On the way down to its minimum position the frictional energy loss is

$$W = \int_0^{x_{\min}} \mu_k mg\cos\alpha dx = \mu_k mg\cos\alpha x_{\min}$$

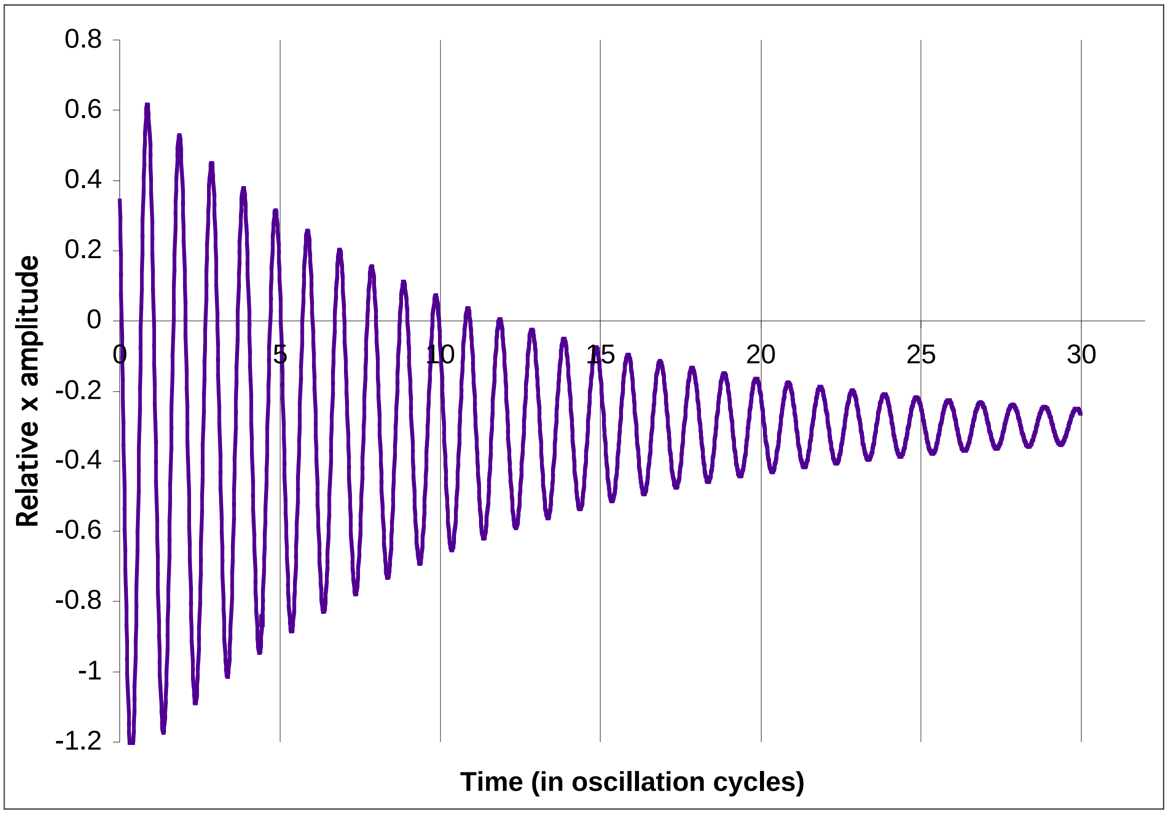
$$= \mu_k mg\cos\alpha\left(\frac{-mg\sin\alpha - \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}\right)$$

On the way back up to the initial point the block will lose the same amount of energy since it travels the same distance, so the kinetic energy will be smaller by an amount

$$\Delta T = 2W = 2\mu_k mg\cos\alpha\left(\frac{-mg\sin\alpha - \sqrt{(mg\sin\alpha)^2 + kmv^2}}{k}\right)$$

(b) The block's velocity is zero twice every oscillation cycle, *i.e.* at a frequency $2\omega/(2\pi)=\omega/\pi=\frac{1}{\pi}\sqrt{\frac{k}{m}}$.

(c)



(d) Because of gravity, the minimum potential energy position of the block is below the initial position (which is the position where the spring force and potential energy are zero). The springs will oscillate about a position below the initial position (*i.e.* from our solutions about the middle point of the oscillation is $x_{middle} = -\frac{mg\sin\alpha}{k}$), so the block as it slows down will be more likely to stop below its initial position.

5) **Fun before television!**

(a) Lets choose an x-y coordinate system in which the positive x direction points vertically down and the positive y direction points horizontally away from the hill. Before the first bounce has a constant horizontal velocity and constant vertical acceleration and the path of the ball as a function of time is

$$y = v\,t\ , \qquad x = \frac{1}{2}\,g\,t^2$$

The parameterization for the slope of the hill is

$$x/y = \tan\theta$$

and the ball will bounce when its coordinates intercept those of the hill, *i.e.* when

$$x_1 = \frac{1}{2}g\left(\frac{y_1}{v}\right)^2 = \frac{1}{2}g\left(\frac{x_1}{v\tan\theta}\right)^2$$

$$\therefore \frac{1}{2}g{x_1}^2 = x_1 v^2 \tan^2\theta$$

$$\therefore x_1 = 2\frac{v^2\tan^2\theta}{g}$$

The distance down the hill for the first bounce is

$$S_1 = \sqrt{{x_1}^2 + {y_1}^2} = \sqrt{{x_1}^2 + \frac{{x_1}^2}{\tan^2\theta}} = \sqrt{1 + \frac{1}{\tan^2\theta}}\,2\frac{v^2\tan^2\theta}{g} = 2\frac{v^2\tan^2\theta}{g\sin\theta}$$

$$= 2\frac{v^2\tan\theta}{g\cos\theta}$$

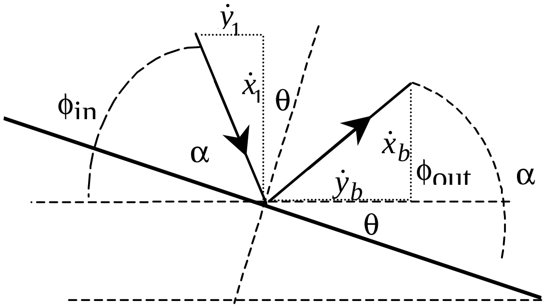
(b) This is pretty much the same as part (a), except a bit messier. First we need the ball's vector velocity $(\dot{x},\dot{y})$ before the bounce.

$$\dot{y}_1 = v\ , \qquad \dot{x}_1 = gt$$

To get the x component we need the time of the bounce:

$$t_1 = \sqrt{\frac{2x_1}{g}} = \sqrt{\frac{2}{g}2\frac{v^2\tan^2\theta}{g}} = \frac{2v\tan\theta}{g}$$

$$\therefore \dot{x}_1 = gt_1 = 2v\tan\theta$$



By energy conservation in an elastic collision the speed (v_1) of the ball must be the same before and after the bounce, so looking at the angles in the bounce, we can see that the velocity components before $(\dot{x},\dot{y})$ and after $(\dot{x}_b,\dot{y}_b)$ the bounce are related by

$$\dot{y}_1 = v_1\cos\phi_{in},\ \ \dot{x}_1 = v_1\sin\phi_{in},\ \ \phi_{in} = \theta + \alpha$$

$$\dot{y}_b = v_1\cos\phi_{out},\ \ \dot{x}_b = v_1\sin\phi_{out},\ \ \phi_{out} = -\theta + \alpha$$

These can be solved in various ways, some of them involving messy trigonometry, but I think the easiest is to think about how each component of $(\dot{x},\dot{y})$ "bounces" separately and then adding them back together, *i.e.*

$$\dot{y}_b = \dot{x}_1\sin(2\theta) + \dot{y}_1\cos(2\theta),\ \ \dot{x}_b = -\dot{x}_1\cos(2\theta) + \dot{y}_1\sin(2\theta)$$

So the path of the ball after the collision is given by

$$y = \dot{y}_b t + y_1, \qquad x = \frac{1}{2}gt^2 + \dot{x}_b t + x_1$$

We can make things easier for ourselves by choosing to redefine our coordinate system such that $x_1 = y_1 = 0$, since the distance bounced can only depend on the initial vector velocity , the acceleration due to gravity, and the angle of the slope. So

$$y = \dot{y}_b t \qquad \Rightarrow \qquad t = \frac{y}{\dot{y}_b}$$

$$x = \frac{1}{2} g t^2 + \dot{x}_b t = \frac{1}{2} g \left(\frac{y}{\dot{y}_b} \right)^2 + \dot{x}_b \frac{y}{\dot{y}_b}$$

Once again the bounce will occur when the ball's coordinates intersect that of the hill (using our redefined coordinate system)

$$\therefore y \tan\theta = \frac{1}{2} g \left(\frac{y}{\dot{y}_b} \right)^2 + \dot{x}_b \frac{y}{\dot{y}_b}$$

$$\therefore y = \frac{2\dot{y}_b{}^2}{g} \left(\tan\theta - \frac{\dot{x}_b}{\dot{y}_b} \right)$$

$$= \frac{2\left(\dot{x}_1 \sin(2\theta) + \dot{y}_1 \cos(2\theta)\right)^2}{g} \left(\tan\theta - \frac{-\dot{x}_1 \cos(2\theta) + \dot{y}_1 \sin(2\theta)}{\dot{x}_1 \sin(2\theta) + \dot{y}_1 \cos(2\theta)} \right)$$

$$= \frac{2\left(2v \tan\theta \sin(2\theta) + v \cos(2\theta)\right)^2}{g} \left(\tan\theta - \frac{-2v \tan\theta \cos(2\theta) + v \sin(2\theta)}{2v \tan\theta \sin(2\theta) + v \cos(2\theta)} \right)$$

$$= \frac{2v^2}{g} \tan\theta \left(1 + 2\sin^2\theta\right)$$

So distance travelled along the slope on the second bounce is

$$S2 = \sqrt{x^2 + y^2} = \sqrt{\tan^2\theta + 1}\, y = \frac{y}{\cos\theta}$$

and the ratio compared to the first bounce is

$$\frac{S2}{S1} = \frac{y}{S1 \cos\theta} = \frac{g}{2v^2 \tan\theta} y = \frac{g}{2v^2 \tan\theta} \frac{2v^2}{g} \tan\theta \left(1 + 2\sin^2\theta\right)$$

$$= \left(1 + 2\sin^2\theta\right)$$

This does not depend on *v* or g; it is 1 when the slope is near horizontal and goes to infinity when the slope is near vertical, both of which make sense since in the former case the ball picks up very little gravitational energy so every bounce should be the same length, and in the latter case the ball picks up a lot of gravitational energy before bouncing.

6) Soup's on! Roll over Galileo!

(a) I practically choked on my breakfast cereal when I read the magazine's explanation in 1996. One of the most famous experiment's in all of science is Galileo dropping balls of different weights from the Leaning Tower of Pisa to show that gravitational acceleration is independent of mass. (For a virtual version of this experiment, see http://www.pbs.org/wgbh/nova/pisa/galileo.html. For a more modern version, check out the video of David Scott dropping a hammer and a feather on the moon at http://cass.jsc.nasa.gov/expmoon/Apollo15/apo15g.avi.) Whether Galileo actually dropped balls from the Tower is somewhat controversial, but we know for sure that Galileo studied acceleration of gravity using balls rolling down inclined planes. From Newton's Laws (of Gravity and Motion) I knew heavier cans should not roll faster, except for effects such a friction and air resistance.

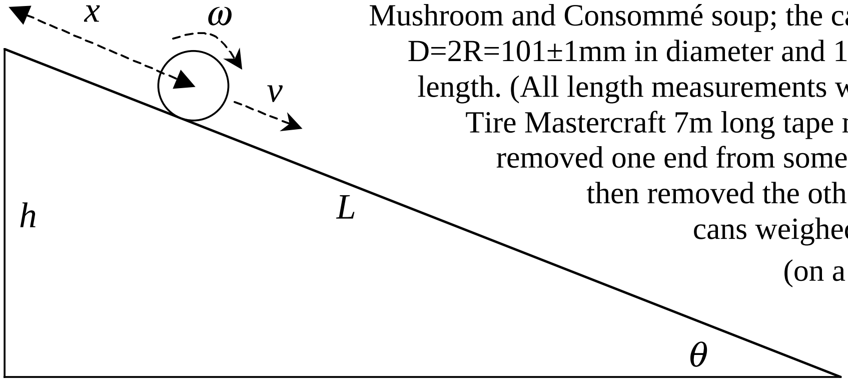
Jumping up from the table, I immediately took a bookshelf (825±10 mm long) and put one end on some books. I took some full cans and empty cans (from our recycling bin) and quickly established that full cans do roll faster than empty cans, but it has nothing to do with how heavy they are. A full 284ml can rolls just as fast as a full 796ml can; similarly a small empty can rolls at about the same speed as a large empty can. The difference must have to do with how the mass is distributed, not how much their is.

I must admit I find it hard to believe that Aristotle actually thought acceleration was proportional to mass. Yes, a flat piece of paper and a rock do fall at very different speeds, but all you need to do is drop a small pebble and a large rock to see that acceleration isn't even close to being proportional to mass. (Even a crumpled up piece of paper hits the ground at almost the same time as a rock if dropped from chest height.) Aristotle was not an idiot (e.g. see http://www.batesville.k12.in.us/Physics/PhyNet/AboutScience/WasAristotle.html), but in this case he seems to have created a classic example of how "common sense" can produce spectacularly wrong but long lasting dogma.

Note: The answers I give below is far longer and more detailed than we expect from you, and even so I have left out much of the details of what I did.

(b) For this problem set, I redid my 5 year old experiment more carefully.

i) In order to measure the acceleration I used 796mL cans of Campbell's Cream of Mushroom and Consommé soup; the cans were D=2R=101±1mm in diameter and 118±1mm in length. (All length measurements with a Canadian Tire Mastercraft 7m long tape measure.) I removed one end from some cans, made soup, then removed the other end. The empty cans weighed m_{empty}=70±5g (on a kitchen scale) and the full cans weighed m_{full}=0.9±0.1kg. (The kitchen scale only went to 250g, so I had to empty the can and measure the contents in several portions which is why the error is large. I roughly checked the calibration of the scale using some water, but the quoted uncertainties are based on how reproducible my measurements were.)



After playing around with various cans and ramps ("playing around" is an essential part of good experimental technique – it is how you get a feel for how to do things), I decided to use a ramp made from a L=825±10mm long Ikea wooden bookshelf resting on VHS video cassettes. I used the cassettes instead of books since they were all the same size (each about 30mm thick). I used this ramp because it was straight, wide, not as slippery like laminate surfaces (so the cans always rolled instead of starting to slide), and it was short enough so I could release the can with one hand and catch with the other. (This last point saved me a lot of time since I didn't have to get up to fetch the can after each run.)

When I raced two cans, the full Cream of Mushroom can beat the empty can by 10±2cm, 8±3cm, 13±3cm for ramp heights of *h* = 120±5mm, 60±5mm, and 182±5mm respectively. I found I could not release the cans simultaneously and see their separation accurately at the bottom (I was alone sine my kids lost interest after about 30minutes), so I decided it was better for me to time the cans one at a time with the stop watch function on my $30 Casio AQ-140 wrist-watch. After more playing around the best method seemed for me to hold the watch in my right hand which also blocked the can at the top of the ramp. I would start the watch with my thumb while raising my hand which released the can. I would stop the watch when the can hit my hand at the bottom of the ramp. I hoped that if I was consistent my reaction times would mostly cancel out.

In order to have a consistent set of data (e.g. what would happen if my reaction times changed) did runs alternating an empty can, a full Cream of Mushroom can, and a full Consommé can so I would have the data for part (c) as well. I rolled each can about 10 times for a given ramp height. I did this for 3 ramp heights (*h*=120±5, 60±5, and 182±5mm), before realizing that my timings depended on where I was looking when I released and caught the cans, and also that by lifting my hand up to release the can, I might sometimes give the can a little push. I decided to continuously look at my catching hand from release to catch, and I pulled my hand away from the can and up to avoid pushing the can (although there is still the possibility of air movements "pulling" the empty can after my hand).

I started again and did a consistent set of measurements ("Can Runs") which are given in a table at the end of this solution set. The times (in seconds) are given for 5 runs for each type of can (Empty, Cream of Mushroom, Full Consommé). The data are presented in the order I took them, but notice that I did not just steadily increase (or decrease) the height of the ramp, to avoid the possibility of a correlated time bias (e.g. maybe I got tired and this affected how I measured the time). I was actually quite pleasantly surprised by how reproducible my measurements were; I didn't realize my reflexes are consistent to 0.1s.

If constant forces (e.g. gravity) on the cans are dominant, then the mean acceleration of the cans is just given by

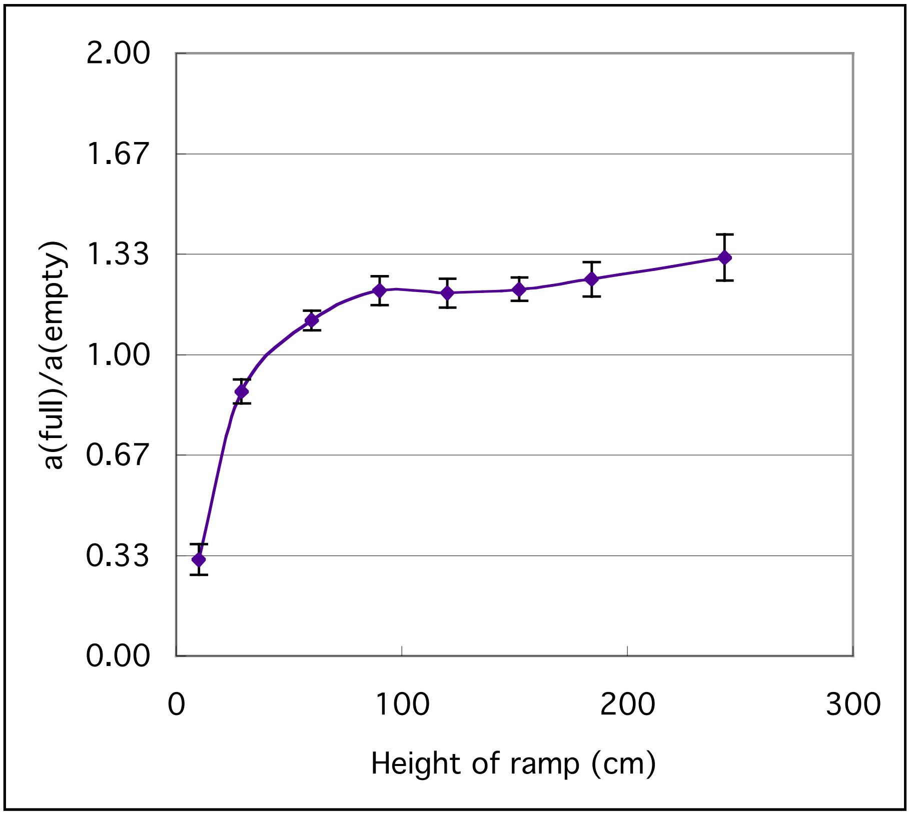
$$a = \frac{2L}{t^2}$$

So since L is a fixed distance,

$$\frac{a_{full}}{a_{empty}} = \left(\frac{t_{empty}}{t_{full}}\right)^2$$

The date for the full and empty cream of mushroom soup is summarized in the table and chart below:

h (in videos)	**h (in mm)**	**σ(h)**	t_{empty}	σ(t_{empty})	t_{full}	σ(t_{full})	$\frac{a_{full}}{a_{empty}}$	σ($\frac{a_{full}}{a_{empty}}$)	**Corrected**
0	10	2	4.72	0.07	8.35	1.33	0.32	0.05	0.30
1	29	2	3.02	0.05	3.22	0.13	0.88	0.04	0.87
2	60	2	2.17	0.03	2.05	0.05	1.11	0.03	1.13
3	90	3	1.83	0.06	1.66	0.03	1.21	0.05	1.26
4	120	3	1.57	0.03	1.43	0.05	1.20	0.05	1.25
5	152	4	1.43	0.04	1.30	0.03	1.22	0.04	1.28
6	184	4	1.31	0.04	1.17	0.04	1.25	0.06	1.33
8	243	5	1.20	0.03	1.04	0.05	1.32	0.08	1.44
11	337	5	1.02	0.04	0.94	0.03	1.17	0.06	1.25



For shallow ramp heights the full can is slower than the empty can, but for steeper ramps the relative acceleration seems to flatten out approaching about 4/3. If I average the last 5 points, I get

$$\frac{a_{full}}{a_{empty}} = 1.23 \pm 0.05$$

ii) The empty can is a hollow cylindrical (radius *R*) with a moment of inertia:

$$I_{empty} = m_{empty} R^2$$

If the cream soup contents rotate uniformly with the can, then the contents of the can are a solid cylinder with a moment of inertia

$$I_{contents} = \frac{1}{2} m_{contents} R^2$$

The total moment of inertial of the full cream of soup can is the sum of the moments of the metal shell and the contents, *i.e.*

$$I_{full} = \frac{1}{2} m_{contents} R^2 + m_{empty} R^2 = \frac{1}{2}\left(m_{contents} + 2m_{empty}\right) R^2 = \frac{1}{2}\left(m_{full} + m_{empty}\right) R^2$$

where $m_{full} = m_{contens} + m_{empty}$.

There are various ways of calculating the acceleration, but I like to follow the energy. When the can rolls down the ramp (height *h*, length *L*, angle with the floor θ), the can gains both translational and rotational kinetic energy, *i.e.*

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

Since the can rolls without slipping, the angular velocity is $\omega = v/R$, so

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{R^2} = \frac{v^2}{2}\left(m + \frac{I}{R^2}\right)$$

The can's kinetic energy after it has rolled a distance x down the ramp must equal the gravitational potential energy it has lost, *i.e.*

$$\frac{v^2}{2}\left(m + \frac{I}{R^2}\right) = mgx\sin\theta$$

$$\therefore\ x = \frac{v^2}{2mg\sin\theta}\left(m + \frac{I}{R^2}\right)$$

$$\therefore\ \frac{dx}{dt} = \frac{v}{mg\sin\theta}\frac{d^2x}{dt^2}\left(m + \frac{I}{R^2}\right)$$

but $v \equiv \frac{dx}{dt}$, so

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{g\sin\theta}{\left(1 + \frac{I}{mR^2}\right)}$$

$$\frac{a_{full}}{a_{empty}}=\frac{\left(1+\frac{I_{empty}}{m_{empty}R^2}\right)}{\left(1+\frac{I_{full}}{m_{full}R^2}\right)}=\frac{\left(1+\frac{m_{empty}R^2}{m_{empty}R^2}\right)}{\left(1+\frac{\frac{1}{2}\left(m_{full}+m_{empty}\right)R^2}{m_{full}R^2}\right)}=\frac{4}{\left(3+\frac{m_{empty}}{m_{full}}\right)}$$

So in the limit where we ignore the weight of the empty can compared to the full can, we have

$$\frac{a_{full}}{a_{empty}}\xrightarrow[\frac{m_{empty}}{m_{full}}\to 0]{}\frac{4}{3}=1.333$$

or using my measurements for my cans

$$\frac{a_{full}}{a_{empty}}=\frac{4}{\left(3+\frac{70\pm 5g}{0.9\pm 0.1kg}\right)}=1.300\pm 0.004$$

(Note: If you solve the torques with respect to the point of contact and you use the wrong moment of inertia - the one with respect to the center of the can- you get a fraction for the two accelerations that is really close to my experimental result: 1.23. This is a coincidence.)

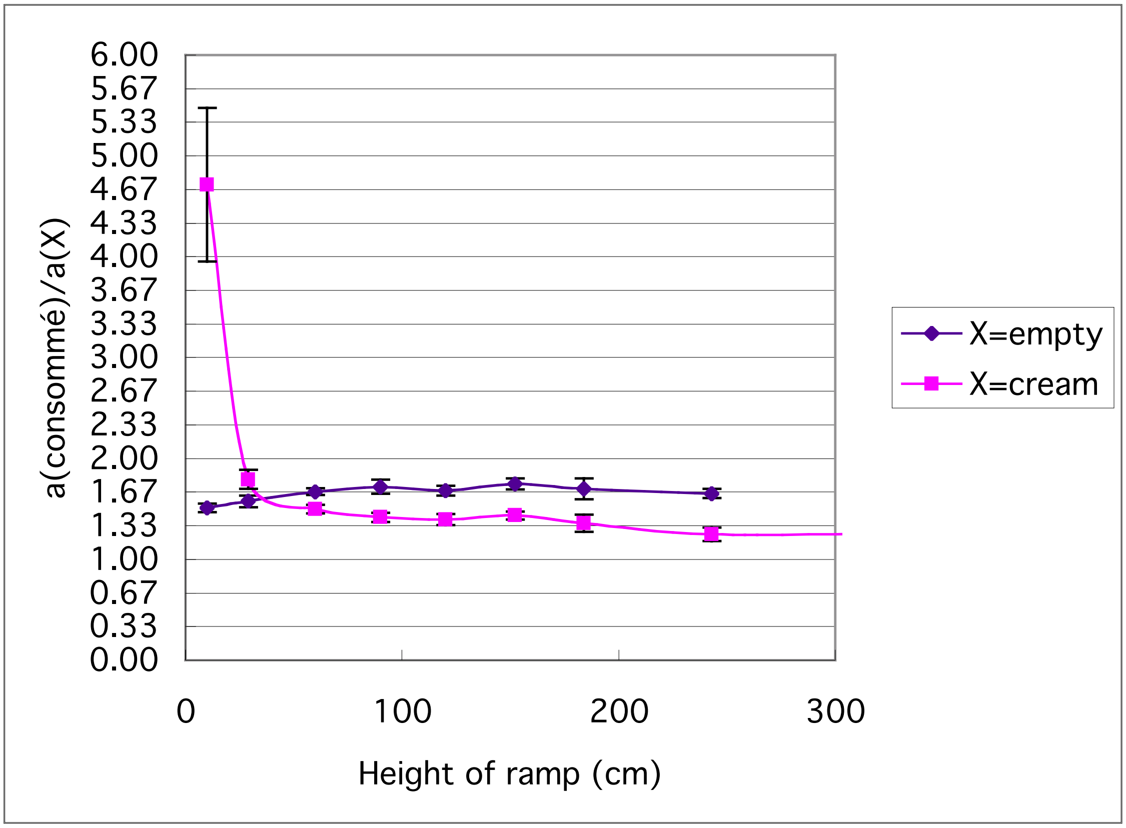
iii) Our theoretical calculation is very close to the experimentally observed value for larger slopes, but several factors could be responsible for the difference and the odd results for small slopes: friction, a constant offset in my timings, air resistance.

Air resistance would have a larger affect at higher speeds, so we would expect it to affect the higher ramp data more, which is not what we see.

I investigated the vary large variations or the shallowest slope (h=10mm, which had no videos and only a thin book propping up the ramp), by trying another can. This can rolled slowly in jerks and almost always stopped and did not reach the bottom of the slope. A little playing around showed that this can preferred to rest in one orientation, apparently because its centre of gravity did not lie on its axis. Since mushroom soup has chunks, this is easy to understand if the mushroom chunks were resting on one side inside the can. The original can probably had a smaller imbalance which was enough to affect its results.

(c) The data at the end of the solution set also includes data for a consommé can.

i) The consommé was clearly the fastest. The plot below shows the data for the relative accelerations of the consommé compared to both the Cream of Mushroom and the empty can.



If we again only consider the last 5 data points, then the average measured ratios are

$$\frac{a_{consommé}}{a_{empty}} = 1.66 \pm 0.09$$

$$\frac{a_{consommé}}{a_{cream\ \ of\ \ mushroom}} = 1.35 \pm 0.08$$

ii) The consommé flows freely inside the can. If friction and viscosity were zero, the can would roll down the slope but the consommé would not rotate within the can. This means that almost all the gravitational potential energy gained by the can would go into translational, not rotational, motion so it would roll with maximum possible acceleration.

$$a_{consommé} = \frac{g\sin\theta}{\left(1+\frac{I_{consommé}}{m_{consommé}R^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{m_{empty}R^2}{m_{consommé}R^2}\right)} = \frac{g\sin\theta}{\left(1+\frac{m_{empty}}{m_{consommé}}\right)}$$

$$\frac{a_{\text{consommé}}}{a_{empty}} = \frac{2}{\left(1+\frac{m_{empty}}{m_{\text{consommé}}}\right)} \xrightarrow[\frac{m_{empty}}{m_{\text{consommé}}}=0]{} 2$$

$$\frac{a_{\text{consommé}}}{a_{cream}} = \frac{\left(1+\frac{\frac{1}{2}\left(m_{full}+m_{empty}\right)}{m_{full}}\right)}{\left(1+\frac{m_{empty}}{m_{\text{consommé}}}\right)} = \frac{\left(\frac{3}{2}+\frac{m_{empty}}{m_{full}}\right)}{\left(1+\frac{m_{empty}}{m_{\text{consommé}}}\right)} \xrightarrow[\frac{m_{empty}}{m_{\text{consommé}}}=0]{} \frac{3}{2}$$

So data indicate the consommé is not quite as fast as our model, which is reasonable since the consommé undoubtedly does rotate somewhat..

Data from Can Runs for Problem 6 (19 December 2001)

height (in videos)	Empty (s)	Cream (s)	Consommé (s)
5	1.45	1.31	1.11
height in mm	1.4	1.33	1.09
152	1.43	1.28	1.06
±	1.48	1.26	1.07
4	1.39	1.3	1.08
3	1.9	1.67	1.39
height in mm	1.82	1.68	1.45
90	1.76	1.63	1.4
±	1.78	1.62	1.36
3	1.88	1.7	1.37
1	3.04	3.39	2.4
height in mm	3	3.15	2.5
29	3.02	3.34	2.45
±	2.95	3.09	2.4
2	3.1	3.15	2.29
6	1.36	1.13	0.95
height in mm	1.32	1.21	0.97
184	1.27	1.19	0.97
±	1.31	1.12	1.05
4	1.27	1.19	1.07
2	2.15	2.05	1.68
height in mm	2.13	2.03	1.67
60	2.18	2.09	1.64
±	2.21	1.98	1.69
2	2.16	2.11	1.7
4	1.57	1.43	1.19
height in mm	1.53	1.41	1.23
120	1.62	1.37	1.22
±	1.59	1.49	1.19
3	1.56	1.47	1.24
8	1.25	1.01	0.93
height in mm	1.17	1.05	0.94
243	1.18	0.99	0.92
±	1.19	1.03	0.94
5	1.2	1.13	0.93
11	0.99	0.91	0.79
height in mm	1.03	0.95	0.82
337	1.06	0.91	0.78
±	1.01	0.91	0.84
5	0.96	0.99	0.9
	1.05	0.96	0.88
0	4.8	8.4	3.77
height in mm	4.78	7.99	3.84
10	4.66	6.85	3.93
±	4.71	10.49	3.95
2	4.66	8.01	3.73