2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 4: Optics

1) Yes, Officer!

If the driver is relativistic, the police officer clearly should issue a ticket, so we can just use the non-relativistic Doppler formula for frequency (f) or wavelength (λ):

$$\frac{f'}{f} = \left(1 + \frac{v}{c}\right) \quad or \quad \frac{\lambda}{\lambda'} = \left(1 + \frac{v}{c}\right)$$

$$\therefore \quad \frac{690nm}{600nm} - 1 = \frac{v}{c}$$

$$\therefore \quad v = 0.15c = 4.5 \times 10^7 \frac{m}{s} = 1.6 \times 10^8 \frac{km}{hour}$$

So the driver is clearly speeding and deserves a ticket. The exact relativistic Doppler shift differs by only a factor of $\sqrt{1 - (v/c)^2} = 0.99$.

2) You push, I'll pull.

The momentum and energy of the photons are

$$p_\gamma = \frac{h}{\lambda} \quad and \quad E_\gamma = \frac{hc}{\lambda}$$

The number of photons produced per second by the sun are

$$N_\gamma = \frac{L_\odot}{E_\gamma} = \frac{L_\odot \lambda}{hc}$$

A dust particle absorbs the momentum of any photon hitting it, so the pressure on a dust particle at a distance r from the sun is the total momentum of the photons per unit area per unit time, *i.e.*

$$P_\odot = \frac{N_\gamma p_\gamma}{4\pi r^2} = \frac{L_\odot h\lambda}{hc4\pi r^2 \lambda} = \frac{L_\odot}{4\pi c r^2}$$

The total light force on a spherical dust grain of radius R is

$$F_L = P_\odot \times Area = P_\odot \pi R^2 = \frac{L_\odot \pi R^2}{4\pi c r^2} = \frac{L_\odot R^2}{4cr^2}$$

The mass of a spherical dust grain of radius R and density ρ is

$$m = \tfrac{4}{3}\pi R^3 \rho$$

The dust grains will be attracted to the sun when the light pressure force is weaker than the gravitational attraction of the sun, *i.e.* when

$$\therefore F_L < F_G$$

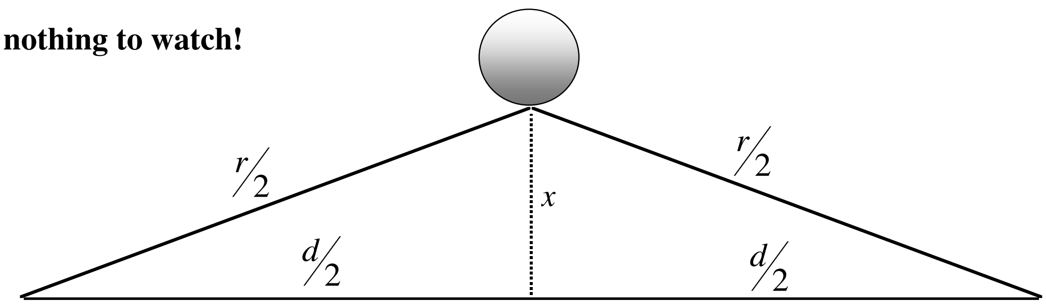
$$\therefore \frac{L_\odot R^2}{4cr^2} < G_N \frac{m M_\odot}{r^2}$$

$$\therefore \frac{L_\odot R^2}{4cr^2} < G_N \frac{\frac{4}{3}\pi R^3 \rho M_\odot}{r^2}$$

$$\therefore R > \frac{3L_\odot}{16\pi c G_N \rho M_\odot} = 1.9\times 10^{-7}\, m$$

So the dust grains must be larger than 0.2μm to fall into the sun. This is why the clouds of gas and small dust are blown away when a star ignites. Another nice example of the effect of radiation pressure on interstellar dust is the Trifid Nebula (http://antwrp.gsfc.nasa.gov/apod/ap990608.html).

3) 500 channels - but nothing to watch!



The distance between me and the TV transmitter are fixed, so the interference indicates that the signal path length is increasing by 8 wavelengths per minute. The path length of the reflected signal is

$$r = 2\sqrt{x^2 + (d/2)^2} = \sqrt{4x^2 + d^2} = \sqrt{4(vt + x_0)^2 + d^2}$$

where t=0 is defined to be when the balloon has a height $x_0 = 20km$. The balloon rises with a constant velocity, $v=dx/dt$, so

$$\frac{dr}{dt} = \frac{d}{dt}\sqrt{4(vt + x_0)^2 + d^2} = \frac{4(vt + x_0)v}{\sqrt{4(vt + x_0)^2 + d^2}}$$

We are interested in the time t=0, when the rate of change of pathlength is

$$\left.\frac{dr}{dt}\right|_{t=0} = \frac{2v}{\sqrt{1 + \left(d/2x_0\right)^2}}$$

The wavelength of the signal is

$$\lambda = \frac{c}{f} = \frac{3\times 10^8 m/s}{57\times 10^6 /s} = \frac{100}{19} m$$

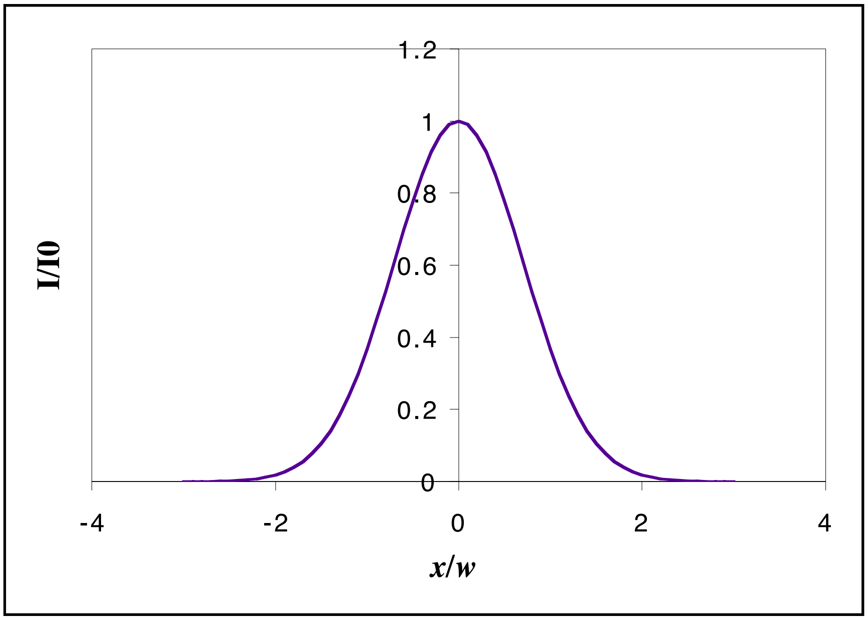
So the balloon is rising with a velocity of

$$v = \frac{1}{2}\left.\frac{dr}{dt}\right|_{t=0}\sqrt{1 + \left(d/2x_0\right)^2} = \frac{1}{2}\frac{8\lambda}{60s}\sqrt{1 + \left(50km/(2\times 20km)\right)^2} = \frac{1}{15s}\left(\frac{100}{19}m\right)\sqrt{1 + \frac{25}{16}} = 0.56 m/s$$

Note: Some people used differences instead of derivatives to get almost the same answer. This works fine in this case since the change in distance is so small compared to the initial distance.

4) An intense experience!

(a)



The width is determined by the parameter *w*.

(b) The beam focuses down because the index of refraction is proportional to the intensity and so is higher nearer to the axis. When passing through varying indices of refraction, light always bends in the direction of higher index of refraction. This follows from Snell's Law or the law of optical distance given as a hint in part (c). In this case there is positive feedback (*i.e.* the light bends towards the region of high index of refraction which increases the intensity there which increases the index of refraction, …), so the light will form a thin beam. After it focuses, it diverges and again focuses and the focusing and diverging will be repeated in the medium, but the net result is a beam. (See, for example http://focus.aps.org/v8/st26.html.)

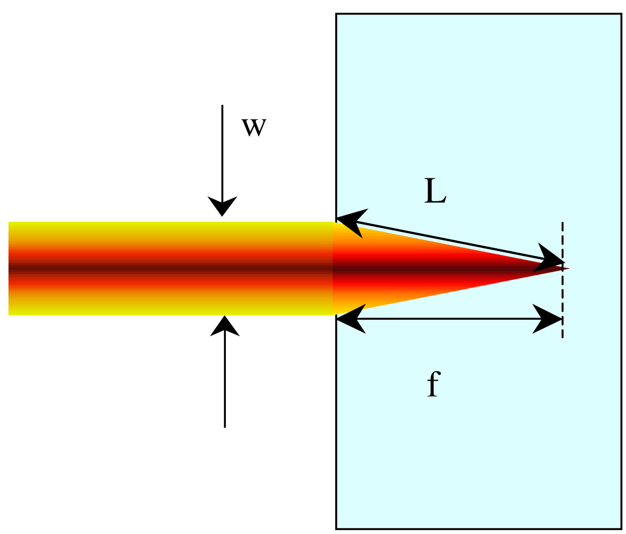
(c) The variation in *n* drops over a distance *w*, so we can make the approximation of triangular focusing

$$L^2 = w^2 + f^2$$

At the edge the index of refraction is n_1, while at the centre it is $n_1 + n_2 I_0$, so using the hint that all the light meeting at a focus will have travelled the same optical distance, we have

$$Ln_1 \sim f\left(n_1 + \frac{n_2}{2} I_0\right)$$

where we use the average index of refraction between the centre and the edge.



$$\therefore \sqrt{w^2 + f^2}\, n_1 \sim f\left(n_1 + \frac{n_2}{2} I_0\right)$$

$$\therefore \left(w^2 + f^2\right) n_1^2 \sim f^2\left(n_1 + \frac{n_2}{2} I_0\right)^2 = f^2\left(n_1^2 + n_1 n_2 I_0 + \frac{n_2^2}{4} I_0^2\right)$$

$$\therefore f \sim \frac{w n_1}{\sqrt{\left(n_1 + \frac{n_2}{4} I_0\right) n_2 I_0}} \xrightarrow[n_1 >> n_2 I_0]{} \sqrt{\frac{n_1}{n_2 I_0}}\, w$$

(d) If $n_2<0$, then the beam will blow up instead of focusing, since the index of refraction will now be lower in the high intensity region.

5) All the world in a drop of water

I used the top part of a CD case to support my drops. The overhead lights in my office provided a nice light source. I had some linear graph paper which had 1mm squares; this made it easy to measure the diameter of the droplets by simply resting the plastic on the graph paper and counting the squares under the drops.

I measured the focal length by seeing how how high above my desk I had to hold the drop to get a sharp image. I initially used a ruler, but then I realized that since I had 2 long parallel fluorescent lights separated by 128±1cm at a distance 203±1cm above my desk, I could get the focal length by measuring the separation of the images on the graph paper and multiplying by 203/128. The drops were rather thick (1.5 mm for the smallest drops, almost 3mm for the biggest diameters), so the thin lens approximation was not very accurate, but probably good enough.

Measuring the magnification was a bit tricky, but first you need to make your drop work as a microscope. Since it is a microscope you have to put your eye very close to the droplet (as the question says quite emphatically). If you don't put your eye close, all you have is a poor magnifying glass. When you do put your eye close, you have to move what your looking at (e.g. your finger nail) nearer and farther until it is in focus. (When it does come into focus the typical reaction, at least for Grade 5 students in my kids' school, is to say "Wow!".)

To measure the magnification for a drop, I first measured the diameter and focal length of a drop using my graph paper. I then rested the CD case on a box of facial tissue which was resting on its end. The box was 23cm long, which is very close to the 25cm standard reference distance, so the drop was held at (approximately) the right distance above the table. I then put a piece of graph paper on the table underneath the drop, and looked at another small piece of graph paper held close underneath the drop. With my eyes very close to the drop I would adjust the small graph paper and move my head until the small graph paper was in focus (often with my eyelashes touching the CD case). I could then see both the graph paper on the table and the magnified small graph paper, and could easily (if not terribly accurately) measure the magnification by comparing the magnified and non-magnified squares. This was hard to do with the smallest drops. It also required a steady hand holding the small piece of graph paper.

Diameter (mm)	Focal Length (mm)	Magnification
3.1±0.3	4.4±0.6	40±5
3.9±0.3	4.8±0.6	32±4
5±1	6.0±0.8	30±3
6±0.4	8.2±0.8	22±3
7.7±0.5	13.5±1	17±2
12.5±0.5	29±3	9±1
18±1	54±6	4±1
5±1	6.0±0.8	30±3
3±0.3	3.5±0.5	40±3

(a) As you see from the above table, I found a 5 mm drop to have a focal length of 6±1mm.

(b) A 5mm drop has a magnification of 30±3.

(c) In general, the larger diameters gave larger focal lengths, larger field of view, and smaller magnification. There are several different kinds of distortion. When looking at graph paper, the lines were more curved near the edges for small drops, but for large drops it was less likely all the field of view would be in focus and the drop was less likely to be circular which lead to more distortion.

One thing I noticed was that difference support plastics made different height drops. CD cases made very high (*i.e.* thick) drops, while the drops were lower on other plastics (*e.g.* overhead transparencies, name tag holders, …). I assume this depended on how hydrophobic the plastic is. I could also make the drops lower by waiting for them to evaporate or poking them with a paper clip. Shorter drops had longer focal lengths and less magnification.

You can see a nice example of a water droplet microscope in the journal Physics Education 36 (March 2001) 97-101 (http://www.iop.org/EJ/S/1/NTO803278/abstract/0031-9120/36/2/301). If you cannot access the IOP website, it is also available (although the beautiful colour photos are only in black and white) from the Applied Spectroscopy Laboratory at the University of Indonesia at http://www.geocities.com/spectrochemical/int_papers/ip34.pdf.