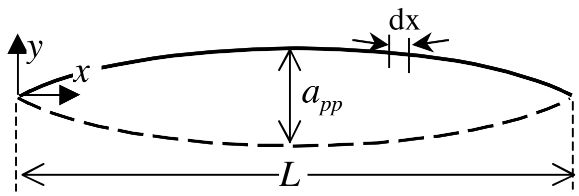
2001-2002 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 5: Electricity and Magnetism

1) Purple Haze!



This can be done using the Lorentz force law, but we'll use Faraday's Law. Assuming the string is oscillating sinusoidally, the induced electromotive force produced by the string sweeping out an area A in the magnetic field B is

$$\varphi = B\frac{dA}{dt} = B\frac{d}{dt}\int_0^L \frac{a_{pp}}{2}\sin(\omega t)\sin\left(\frac{\pi x}{L}\right)dx$$

$$= B\frac{a_{pp}}{2}\omega\cos(\omega t)\frac{L}{\pi}\left[\cos\left(\frac{\pi x}{L}\right)\right]_0^L = -2BLa_{pp}\nu\cos(\omega t)$$

where $\omega=2\pi\nu$ is the radial frequency and ν=330Hz is the frequency of oscillation. The magnitude of the peak e.m.f. is then

$$\varphi_p = \left|-2BLa_{pp}\nu\right| = 2(1.5T)(0.66m)(0.001m)330s^{-1} = 0.7V$$

This is not enough voltage to shock Steve, but is the string temperature safe?

By Joule's and Ohm's Laws, the power produced in a segment, dx, of the string with voltage drop dV across its resistance dR is

$$dP = \frac{(dV)^2}{dR} = \frac{\left(B\frac{a_{pp}}{2}\omega\cos(\omega t)\sin\left(\frac{\pi x}{L}\right)dx\right)^2}{\frac{\rho}{A}dx} = \frac{AB^2a_{pp}^2\omega^2}{4\rho}\cos^2(\omega t)\sin^2\left(\frac{\pi x}{L}\right)dx$$

where A is now the cross-sectional area of the string and ρ is it's resistivity. The average power produced over one complete oscillation of the string is

$$\langle dP\rangle = \frac{AB^2a_{pp}^2\omega^2}{4\rho}\sin^2\left(\frac{\pi x}{L}\right)dx\left\langle\cos^2(\omega t)\right\rangle = \frac{AB^2a_{pp}^2(2\pi\nu)^2}{4\rho}\sin^2\left(\frac{\pi x}{L}\right)dx\frac{\int_0^{2\pi}\cos^2(\omega t)dt}{\int_0^{2\pi}dt}$$

$$= \frac{AB^2a_{pp}^2\pi^2\nu^2}{\rho}\sin^2\left(\frac{\pi x}{L}\right)dx\frac{\left[\frac{1}{2}-\frac{1}{4\omega}\sin(2\omega t)\right]_0^{2\pi}}{2\pi} = \frac{AB^2a_{pp}^2\pi^2\nu^2}{2\rho}\sin^2\left(\frac{\pi x}{L}\right)dx$$

Since heat is being produced, the string will heat up. By symmetry, the middle of the string must be the hottest point, and the net heat flow in the right half of the string (in the figure) is to the right, and the flow in the left half is too the left. The string will heat up until it reaches a steady state where the heat flowing through each segment just matches the joule heating of the part of the string whose heat must flow out through the segment (*i.e.* integrated from the midpoint to the segment):

$$\frac{dQ}{dt} = -kA\frac{dT}{dx} = \int_{L/2}^{x} \langle dP \rangle = \int_{L/2}^{x} \frac{AB^2 a_{pp}^2 \pi^2 \nu^2}{2\rho} \sin^2\left(\frac{\pi x'}{L}\right) dx'$$

$$= \frac{AB^2 a_{pp}^2 \pi^2 \nu^2}{2\rho}\left[\frac{x'}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x'}{L}\right)\right]_{L/2}^{x} = \frac{AB^2 a_{pp}^2 \pi^2 \nu^2}{2\rho}\left[\frac{x}{2} - \frac{L}{4} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right]$$

(k is the thermal conductivity of the string.)

$$\therefore \frac{dT}{dx} = \frac{B^2 a_{pp}^2 \pi^2 \nu^2}{8k\rho}\left[L - 2x + \frac{L}{\pi}\sin\left(\frac{2\pi x}{L}\right)\right]$$

The temperature profile of the string is thus

$$T(x) - T(0) = \int_0^x \frac{dT}{dx'}dx' = \int_0^x \frac{B^2 a_{pp}^2 \pi^2 \nu^2}{8k\rho}\left[L - 2x' + \frac{L}{\pi}\sin\left(\frac{2\pi x'}{L}\right)\right]dx'$$

$$= \frac{B^2 a_{pp}^2 \pi^2 \nu^2}{8k\rho}\left[Lx - x^2 - \frac{L^2}{2\pi^2}\left(\cos\left(\frac{2\pi x}{L}\right) - 1\right)\right]$$

The peak temperature of the string is for x=L/2

$$T(x) = \frac{B^2 a_{pp}^2 \nu^2 L^2}{8k\rho}\left[\frac{\pi^2}{4} + 1\right] + T(0)$$

$$= \frac{(1.5T)^2(0.001m)^2(330Hz)^2(0.66m)^2}{8\left(12\times10^{-6}\Omega\cdot10^{-2}m\right)(80W/m/K)}\left[\frac{\pi^2}{4} + 1\right] + (21K + 273K)$$

$$= 5100K$$

This means that Steve will burn his fingers, and if he tries to keep on playing with an insulated pick the string will melt (the element with the highest melting point, 3800K, is carbon, but a carbon fibre or diamond string would burst into flames first; the metal with the highest melting point, 3695K, is Tungsten, see http://www.webelements.com). I think Steve should keep his guitar out of the CDF magnet.

Alternate shorter approach from Alex:

The area swept out by the string is

$$A = \int_0^{660mm} (1mm)\sin\left(\frac{\pi x}{660mm}\right)dx = 420mm^2$$

and this area is swept out twice per cycle, so the average e.m.f. is

$$\langle \varphi \rangle \cong B\left\langle \frac{dA}{dt} \right\rangle = 1.5T\cdot 2\cdot 330Hz\cdot 420mm^2 = 0.42V$$

The resistance is

$$\langle\varphi\rangle \cong BR = \frac{\rho L}{A} = \frac{\left(12\times10^{-6}\,\Omega\cdot cm\right)66cm}{\pi\left(0.023cm\right)} = 1.9\Omega$$

So the total power is

$$\langle\varphi\rangle \cong P = \frac{V^2}{R} = 93mW$$

Half the power is lost through the end of the string (with most power being produced in the middle). Thus

$$\frac{P}{2} = \frac{\sigma\pi d^2}{4L/2}\Delta T$$

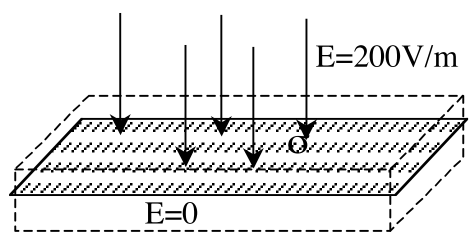
$$\therefore \Delta T = \frac{LP}{\sigma\pi d^2} = 4600K$$

$$\therefore T_{middle} = T_{end} + \Delta T \approx 4900K$$

2) Relatively electric!

(a) People are much better conductors than air, so they short out the field so their whole body is at a constant potential. The energy density of the atmospheric "battery" is so low that only an infinitesimal surface current flows through the body.

(b) The earth is a conductor, so it will have zero field inside it and a charge density on its surface which can be calculated from the electric field at the surface using Gauss's Law. Near the surface of the earth it looks like a plane, so we can just use a Gaussian box with top (and bottom) area A, *i.e.*



$$\frac{q_{insidebox}}{\varepsilon_0} = \vec{E}_{top}\cdot\vec{A}_{top} + \vec{E}_{bottom}\cdot\vec{A}_{bottom} + \vec{E}_{sides}\cdot\vec{A}_{sides}$$

$$\therefore \frac{\sigma A}{\varepsilon_0} = EA + 0\cdot A + E_{sides}A_{sides}\cos\tfrac{\pi}{2}$$

$$\therefore \frac{\sigma}{\varepsilon_0} = E$$

So the surface charge density is

$$\sigma = \varepsilon_0 E = \frac{\left(8.854\times10^{-12}F/m\right)\left(-120V/m\right)}{1.6\times10^{-19}C/e} = -6.6\times10^{6}\left|electron\ \ charges\right|/m^2$$

Note that the surface charge density is negative.

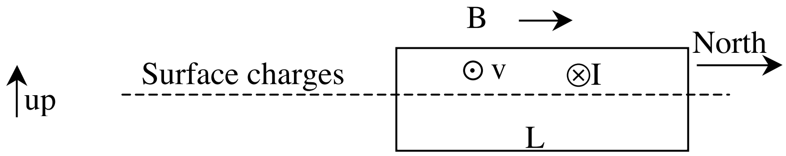
(c)

i) In the rest frame of the earth's surface, the magnetic field generated by the surface charge density is zero, since it isn't moving in this reference frame.

ii) In this frame, the charge is moving with a velocity

$$v = \frac{2\pi R_{\oplus}}{1 day} = \frac{2\pi 6.378\times 10^{6} m}{24 hours \times 3600 s/hour} = 464\frac{m}{s}$$

underneath the observer. This corresponds to a current sheet with a current density σ*v*. By symmetry and the Biot-Savart Law, the magnetic field must run at right angles to the velocity and parallel t to the surface. The direction of the magnetic field can be determined to be from south to north using the right hand rule. so we can use Ampere's Law to calculate the magnetic field:



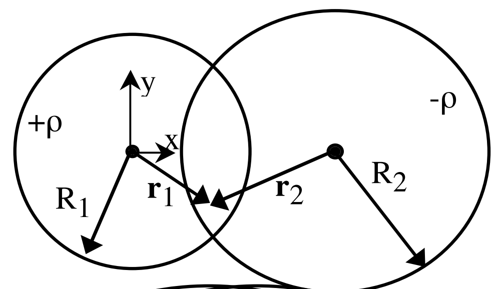
$$\oint_{loop} \vec{B}\cdot d\vec{s} = \tfrac{1}{2}\mu_0 I_{enclosed}$$

$$\therefore BL = \tfrac{1}{2}\mu_0 L\sigma v$$

$$\therefore B = \tfrac{1}{2}\mu_0\sigma v = \tfrac{1}{2}\mu_0\varepsilon_0 E v = \frac{Ev}{2c^2} = \tfrac{1}{2}\beta\frac{E}{c} = \tfrac{1}{2}\frac{(120V/m)(464m/s)}{\left(3\times 10^{8} m/s\right)^2} = 0.3pT$$

(d) So the magnetic field is the right direction, but much too weak.

3) Spheres within spheres



(a) The electric field in the region of overlap is

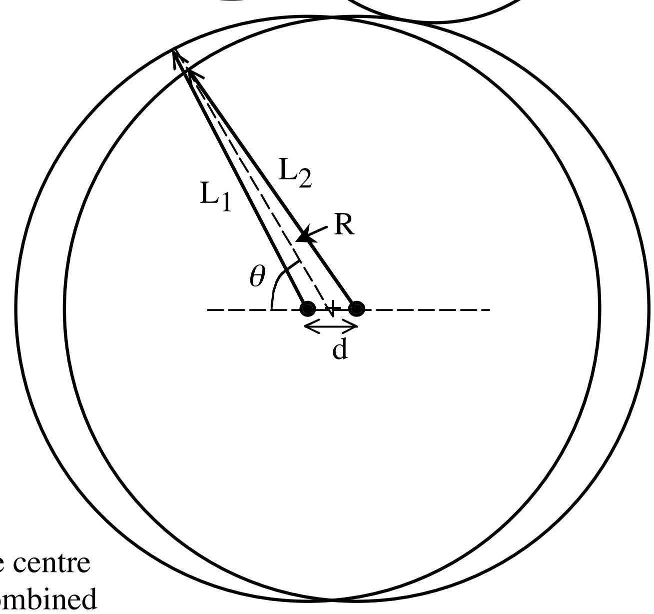
$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\rho\mathbf{r}_1}{3\varepsilon_0} + \frac{-\rho\mathbf{r}_2}{3\varepsilon_0} = \frac{\rho}{3\varepsilon_0}(\mathbf{r}_1 - \mathbf{r}_2)$$

Choosing sphere 1 as the origin, and defining the x axis to be the direction from the centre of sphere 1 to the centre of sphere 2, then

$$\mathbf{E} = \frac{\rho}{3\varepsilon_0}(x_1 - x_2, 0, 0)$$

$$\therefore \mathbf{E} = \frac{\rho}{3\varepsilon_0} d\ \mathbf{x} ✓$$

(b)



i) The charges cancel where the spheres overlap, so the surface charge density is just σ(*θ*)= ρ *t*(*θ*), where *t*(*θ*) is the thickness of the uncancelled charge layer at the surface. The maximum charge thicness is obviously at *t*(0)=d, σ_0=ρd.

Using the cosine rule and the fact that the centre of each sphere is ofset by ±d/2 from their combined centre, we can calculate the thickness in the limit where d<<R, *i.e.*

$$
\begin{aligned}
t(\theta) &= L_2(\theta) - L_1(\theta) \\
&= \sqrt{(L_2(\theta))^2} - \sqrt{(L_1(\theta))^2} \\
&= \sqrt{R^2 + \left(\frac{d}{2}\right)^2 - \frac{2Rd}{2}\cos\theta} - \sqrt{R^2 + \left(\frac{d}{2}\right)^2 - \frac{2Rd}{2}\cos(\pi - \theta)} \\
&\cong R\left(1 + \frac{d}{R}\cos\theta\right) - R\left(1 - \frac{d}{R}\cos\theta\right) \\
&= d\cos\theta
\end{aligned}
$$

where we have used the approximation that $\sqrt{1+\varepsilon} \cong 1 + \frac{1}{2}\varepsilon$ when ε<<1. So the surface charge density is

$$\sigma = \rho \;\; \mathrm{d}\cos\theta \;\; = \;\; \sigma_0 \cos\theta$$

and the electric field strength inside the (overlapped) sphere is

$$\therefore \mathbf{E} = \frac{\sigma_0}{3\varepsilon_0} \;\; \hat{\mathbf{x}}$$

ii) Outside the spheres, this charge configuration is just an electric dipole with dipole moment

$$\mathrm{p} = Q_{sphere}\mathrm{d} = \tfrac{4}{3}\pi R^3 \rho \mathrm{d} = \tfrac{4}{3}\pi R^3 \sigma_0$$

The electric potential outside the spheres is just the dipole potential

$$V(r) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\varepsilon_0 r^3} = \frac{\mathrm{p}}{4\pi\varepsilon_0 r^2}\cos\theta$$

so at the surface of the sphere

$$V(r) = \frac{\frac{4}{3}\pi R^3 \sigma_0}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{R\sigma_0}{3\varepsilon_0}\cos\theta = \frac{R\rho \mathrm{d}}{3\varepsilon_0}\cos\theta = \frac{R\sigma}{3\varepsilon_0}$$

4) Does it (anti)matter?

Conduction electrons are free to move, so if they are at rest they must feel no net force. In a piece of copper on the surface of the earth, the conduction electrons feel the force of gravity, but they don't all fall out the bottom of the copper, so there must be a force balancing gravity. This force is an equal and opposite electrostatic force generated by the electrons themselves sagging under the force of gravity. *i.e.* They start to fall but this very quickly causes the bottom of the copper to become slightly negatively charged relative to the top, and the electrons will stop moving when the charge imbalance is such that it exactly balances gravity everywhere inside the conductor. The electrons natural arrange themselves for this to happen, and this is just the same explanation why the electric field inside a conductor is always zero (except they never mention that the effect of gravity is to create a tiny non-zero electrostatic field).

For a positron, the electrostatic force is the equal and opposite to that on a electron, but the gravitational force is equal and in the same direction (according to General Relativity), so if the net force on a conduction electron is zero, the net force on a positron must be

$a_{positron} = 2g = 19.6$m/s/s straight down.

(g is the standard acceleration of gravity at the earth's surface.)

5) Refrigerator Art

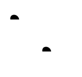
Playing around with two fridge magnets, I noticed that if you stick them together like this (where the white area is the printed face and the dark area is the magnetic part),

and then gently pull them like this, they

make a "clack, clack, clack" sound. Studying this effect more closely it is clear that what is happening is that they start to repel each other as I pull them

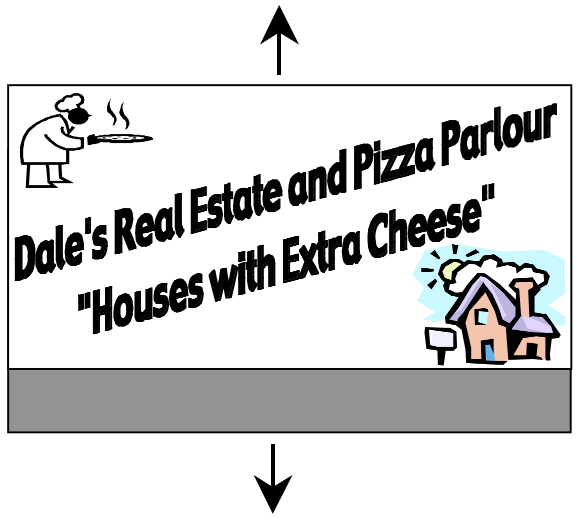
and then they are attracted,

and then repelled, and so on,

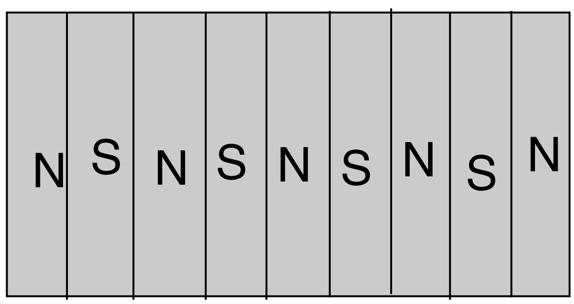


and each time they are attracted together they make a nice "clack!". This only happens when I pull them along their long axis, *i.e.* like this

not when I pull them like this



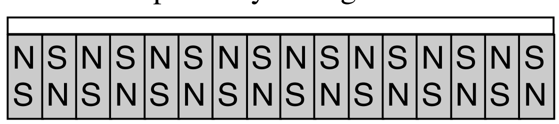
This must mean that fridge magnet must consist of many magnetic domains (essentially little dipole magnets) whose North and South poles must be arranged like



(I can't tell which is north and which is south, but they must be alternating.). The positions where the fridge magnets were attracted together were at 5, 9, 13, and 17 mm, so the width of the domains is about $4\,^{1}/_{2}$ mm. (Your magnets may differ, although all the ones I tried seemed similar.)

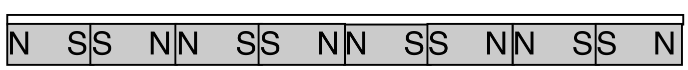
Looking from the side, the domains are probably arranged like this

(A)



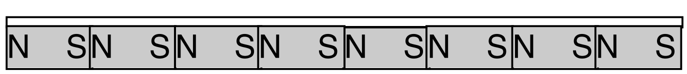
or less probably like this

(B)

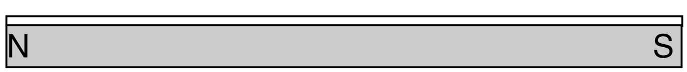


It cannot be like this

(C)

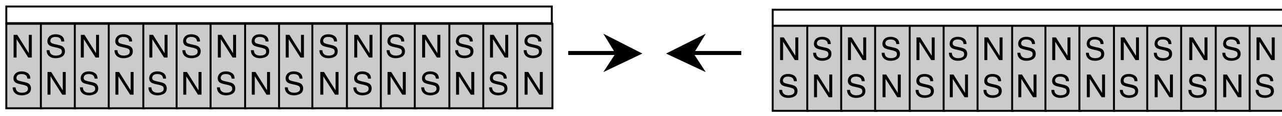


since that us equivalent to this

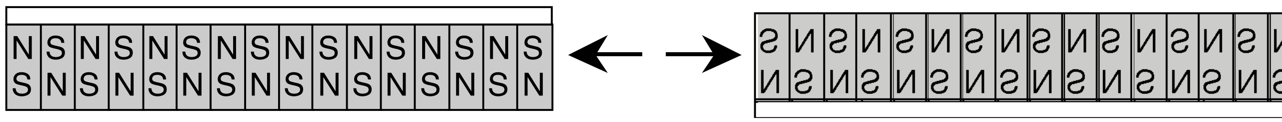


(assuming all domains are equal strength magnets) which would not give the correct "clack, clack behaviour.

To check that arrangement (A) is correct, I held the ends of the fridge magnets close to each other. They either attracted or repelled, *e.g.*



When I flipped them over, they did the opposite *e.g.*



(This took some care to see since the forces are weak and the main observable affect is not the repulsion but the fact that the magnets tend to push to one side to try to align North to South poles.) These observations are not consistent with arrangement (B), in which case the force should not change direction.