2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 4: Waves and Optics

1. In equilibrium, the net force on the ice cube is

$$F_{net} = -W_i + B = 0 \to W_i = B \to \rho_{ice} V_{ice} g = \rho_{water} V_{water} g \ , \quad (1)$$

where W_i (B) is the gravitational (buoyancy) force on the cube, $V_{ice} = L^3$ (V_{water}) is the volume of the ice (water displaced by the cube), and g is the acceleration due to gravity. When the ice cube is raised by a small amount x, the net force is

$$F_{net} = -W_i + B' \to Ma = -\rho_{ice} V_{ice} g + \rho_{water}(V_{water} - L^2 x) g = -\rho_{water} L^2 g x \ , \quad (2)$$

where $M = \rho_{ice} V_{ice}$ and $a = \frac{d^2x}{dt^2}$ are the mass and acceleration of the cube. Eq. (1) was used to simplify Eq. (2). The latter reduces to

$$\frac{d^2x}{dt^2} + \frac{\rho_{water} L^2 g}{\rho_{ice} L^3} x = \frac{d^2x}{dt^2} + \omega^2 x = 0 \ , \quad (3)$$

which is the equation for simple harmonic oscillation. The angular frequency is $\omega = \sqrt{\frac{\rho_{water} g}{\rho_{ice} L}} = \sqrt{\frac{1000 \cdot 9.8}{900 \cdot 1}} \approx 3.3 \ \mathrm{s}^{-1}$ and the frequency is $\nu = \frac{\omega}{2\pi} \approx 0.53 \ \mathrm{s}^{-1}$.

2. At $t = 0$, the source S emits a spherical wave front with a radius that increases as $R = v_s t$. We are interested in the intensity of the sound waves reaching a detector D at a distance $r = R - vt$ in front of S. The energy flowing through area A during time T is $E_A = \frac{PT}{A}$, where T is the period of the sound wave. Since S is moving towards D, the period of the waves T' is Doppler shifted such that $\frac{1}{T'} = \frac{1}{T} \frac{v_s}{v_s - v}$. The above equations reveal that

$$r = R - vt = v_s t - vt \to t = \frac{r}{v_s - v} \to R = \frac{v_s}{v_s - v} r \ , \quad (4)$$

$$E_A = \frac{PT}{4\pi R^2} = \frac{PT(v_s - v)^2}{4\pi v_s^2 r^2} \ . \quad (5)$$

The intensity, or energy flowing through unit area per unit time, at D is

$$I = \frac{1}{T'} E_A = \frac{P}{4\pi r^2} \frac{v_s - v}{v_s} \ . \quad (6)$$

3. Light travelling from air to glass, and then from glass to air, refracts according to $n_1 sin\theta_1 = n_2 sin\theta_2$ and $n_2 sin\theta_2 = n_1 sin\theta_3$, where $\theta_1 \equiv \theta$ and $n_2 = n$. (See Figure 1.) These equations indicate that $\theta_1 = \theta_3$. The beam enters and exits from the glass at the same angle to the surface normal. From the geometry shown in Figure 1, $cos\theta_2 = \frac{L}{D}$ and $sin(\theta - \theta_2) = \frac{x}{D}$. The latter implies that $x = D sin(\theta - \theta_2)$. The beam will be shifted by a distance x with respect to the direction of entry as long as $\theta = \theta_2 \neq 0$.

$$x = D sin(\theta - \theta_2) = \frac{L}{cos\theta_2}[sin\theta cos\theta_2 - cos\theta sin\theta_2] \ , \quad (7)$$

$$\frac{x}{L} = sin\theta - tan\theta_2 cos\theta \to tan\theta_2 = \frac{sin\theta - x/L}{cos\theta} \quad (8)$$

The index of refraction of the medium is

$$n = n_1 \frac{sin\theta}{sin\theta_2} = n_1 \frac{sin\theta}{sin[tan^{-1}(tan\theta - \frac{x}{Lcos\theta})]} \ . \quad (9)$$

4. To find the maxima on the z-axis, set the path difference of the waves originating from the sources equal to an integral number of wavelengths. (See Figure 2.)

$$r_2 - r_1 = m\lambda \rightarrow \sqrt{d^2 + z^2} - z = m\lambda \quad \rightarrow z = \frac{d^2 - m^2\lambda^2}{2m\lambda}\ , \tag{10}$$

where $m = 1, 2, \ldots$. The above must be restricted so that $z > 0$. This implies that

$$d^2 - m^2\lambda^2 \quad = \quad (d - m\lambda)(d + m\lambda) > 0 \rightarrow m < \frac{d}{\lambda}\ . \tag{11}$$

There are a finite number of maxima on the z-axis corresponding to $m = 1, 2, \ldots, int(\frac{d}{\lambda})$, where $int(x)$ is the largest integer less than x. Consider the spacing of the maxima along a line $z = D$ that is perpendicular to the z-axis. The maxima are located at $dsin\theta = n\lambda$, where n is an integer. For large D, $dsin\theta = d\frac{y}{\sqrt{y^2+D^2}} \approx \frac{dy}{D} = n\lambda \rightarrow y = \frac{n\lambda D}{d}$. The spacing of the maxima in the direction orthogonal to $\hat{z}$ increases with D. For large enough D, no maxima will be found along the z-axis. The minima appear at

$$r_2 - r_1 \quad = \quad (m + \frac{1}{2})\lambda \rightarrow z = \frac{d^2 - (m+\frac{1}{2})^2\lambda^2}{2(m+\frac{1}{2})\lambda}\ . \tag{12}$$

The restriction $z > 0$ yields minima corresponding to $m = 0, 1, 2, \ldots, int(\frac{d}{\lambda} - \frac{1}{2})$. The minimum closest to the sources is z_{min} such that $m = int(\frac{d}{\lambda} - \frac{1}{2})$. The intensity at $z = z_{min}$ is

$$I(0, 0, z_{min}, t) \quad = \quad \frac{1}{2}\epsilon_0 c|E|^2 = \frac{1}{2}\epsilon_0 c|\frac{E_0}{r_1}e^{i(kr_1 - \omega t)} + \frac{E_0}{r_2}e^{i(kr_2 - \omega t)}|^2 \tag{13}$$

$$= \quad \frac{1}{2}\epsilon_0 c E_0^2(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{2}{r_1 r_2}cos[k(r_1 - r_2)]) \tag{14}$$

$$= \quad \frac{1}{2}\epsilon_0 c E_0^2(\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2}{r_1 r_2})\ , \tag{15}$$

where $r_1 = z_{min}$ and $r_2 = \sqrt{z_{min}^2 + d^2}$.

5. Light that originates from the submerged object O, refracts towards the surface of the water when it enters the air. The object appears higher than it actually is. If your eyes are horizontal with respect to the water's surface, then there is no alteration of the horizontal distance to O. If your eyes are vertical, then there is a difference between the light reaching your left and right eye. This light has been refracted by different amounts. The object looks closer and there is a change in its apparent horizontal distance.