2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 5: Electricity and Magnetism

1. Gauss' law is $q = \epsilon_0 \int \mathbf{E} \cdot d\mathbf{S}$, where $\mathbf{E}$ is the electric field. The integral is over a surface, such as a cylinder, enclosing an amount of charge q. Over the sides of the cylinder, $\mathbf{E} \cdot d\mathbf{S} = 0$. The contributions from the ends of the cylinder give

$$\begin{aligned} q &= \epsilon_0(-E_{Cu}A + E_{Al}A) = \epsilon_0 A(E_{Al} - E_{Cu}) = \epsilon_0 A(\frac{\rho_{Al}I}{A} - \frac{\rho_{Cu}I}{A}) &(1)\\ &= \epsilon_0 I(\rho_{Al} - \rho_{Cu}) = (8.85 \times 10^{-12})(1)(2.8 - 1.7) \times 10^{-8} &(2)\\ &\approx 9.735 \times 10^{-20}\ \mathrm{C} \approx 0.608\ \mathrm{e}\ , &(3)\end{aligned}$$

where A is the area of the cylinder's ends, I is the current flowing through the wire, and E_{Cu} (E_{Al}) is the magnitude of the electric field in the copper (aluminum). The electric field in a wire with a cross sectional area A and resistivity ρ is $E = \frac{\rho I}{A}$. The amount of charge that accumulates at the boundary between the metals is less than e, the charge of an electron. Something is incorrect about the above calculation since under ordinary conditions, fractional charges are not observed in nature.

2. The electric potential between the two parallel metal plates is

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = xE + bE_\kappa + (d - b - x)E = (\frac{b}{\kappa} + d - b)E\ , \qquad (4)$$

where the integral is evaluated along a line between the plates, E is the magnitude of the electric field between the plates, and $E_\kappa = \frac{E}{\kappa}$ is the field inside the dielectric slab when it is within the capacitor. According to Gauss' law, $q = \epsilon_0 AE$ and $E = \frac{q}{\epsilon_0 A}$, where q is the charge on the plates. The capacitance with the dielectric slab present is

$$C_\kappa = \frac{q}{V} = \frac{q}{(\frac{b}{\kappa} + d - b)E} = \frac{\epsilon_0 A}{\frac{b}{\kappa} + d - b}\ . \qquad (5)$$

Set $b = 0$ to find $C = \frac{\epsilon_0 A}{d}$, the capacitance of the empty capacitor. If $C_\kappa = 2C$, then $\frac{\epsilon_0 A}{\frac{b}{\kappa}+d-b} = 2\frac{\epsilon_0 A}{d}$ implies $\kappa = \frac{2b}{2b-d}$. We must also have that $\kappa > 0$ and $b \leq d$. The restrictions on κ and b are $\kappa = \frac{2b}{2b-d}$ and $\frac{d}{2} < b \leq d$.

3(a) Ampere's Law is $\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 IN$, where $\mathbf{B}$ is the magnetic field, μ_0 is the permeability, and IN is the total current contained in the N turns of wire. By symmetry, the field inside the toroid is $\mathbf{B} = B(r)\hat{\theta}$, where $B(r)$ is given by

$$\int \mathbf{B} \cdot d\mathbf{l} = \int B dl = B \int dl = B 2\pi r = \mu_0 IN \rightarrow B(r) = \frac{\mu_0 IN}{2\pi r}\ . \qquad (6)$$

The integral is performed along a circular path of radius r.
(b) The force on the electron is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \rightarrow F = qvB = \frac{mv^2}{R} \rightarrow RB = \frac{mv}{q}\ , \qquad (7)$$

where R is the radius of curvature of the electron's path and $B = B(r)$ from part (a). Since $\mathbf{F}$ and the direction of the electron's motion are always orthogonal, the work done on the electron by $\mathbf{B}$ is $\mathbf{F} \cdot \mathbf{x} = 0$. The speed of the electron v_0 does not change and $RB(r) = R\frac{\mu_0 IN}{2\pi r}$ remains constant. The radii of curvature at two different distances from the centre of the toroid are related via $\frac{R_1}{r_1} = \frac{R_2}{r_2} \rightarrow R_2 = \frac{r_2}{r_1}R_1$. This shows that if $r_2 < r_1$, then $R_2 < R_1$. The radius of curvature becomes smaller as r decreases. The electron executes a curved path as indicated in Figure 3.

4. Find the magnetic field due to the current in L via the Biot-Savart law

$$dB = \frac{\mu_0}{4\pi}\frac{I sin\theta dl}{r^2} . \tag{8}$$

By symmetry, only the z component of $\mathbf{B}$ is nonzero at points on the z-axis.

$$dB_z = dB cos\alpha = \frac{\mu_0}{4\pi}\frac{I sin(\frac{\pi}{2}) dl \frac{a}{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I a dl}{r^3} \tag{9}$$

$$B_z = \int dB_z = \frac{\mu_0}{4\pi}\frac{Ia}{r^3}\int dl = \frac{\mu_0}{4\pi}\frac{Ia(2\pi a)}{(z^2+a^2)^{3/2}} = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}} . \tag{10}$$

Assume that the above magnetic field is constant across the area of S. The magnetic flux Φ through S is

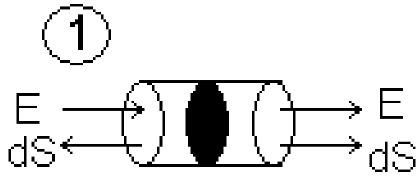
$$\Phi = \mathbf{B} \cdot \mathbf{A} = B\pi b^2 = \frac{\mu_0 \pi I a^2 b^2}{2(z^2+a^2)^{3/2}} . \tag{11}$$

The electromotive force (voltage) in S is

$$\varepsilon(t) = -\frac{\partial \Phi}{\partial t} = -\frac{-3\mu_0 \pi I a^2 b^2 z}{2(z^2+a^2)^{5/2}}\frac{dz}{dt} = \frac{3\mu_0 \pi I a^2 b^2 z(t) v_z(t)}{2(z(t)^2+a^2)^{5/2}} . \tag{12}$$

Following its release, the small loop accelerates towards the large loop. The former's position and velocity are $z = z(t) = z_0 - \frac{1}{2}gt^2$ and $\mathbf{v} = v_z(t)\hat{z} = -gt\hat{z}$. The net force on the current in S due to L's magnetic field is zero. The time dependent current in S is $I_S(t) = \frac{\varepsilon(t)}{R}$ and it flows clockwise to produce a magnetic field that opposes the field of L.

5. A motor converts electrical energy into mechanical energy. The design in Figure 5 consists of a wire loop L connected to a battery. The magnetic field due to the magnet M exerts a force on the current in L. If the current always flows in the same direction, L will rotate once or twice until it is horizontal with respect to the field. The net force on L is then zero. The loop may also twitch in a random way and some peculiar motions are possible. If the current flows one direction during a half rotation of L and changes direction during the next half rotation, then it is possible for a continuous net force to be applied to the current carrying wire. This depends on the nature of the contact between L and the wire holder. The loop will rotate in one direction until the battery is exhausted.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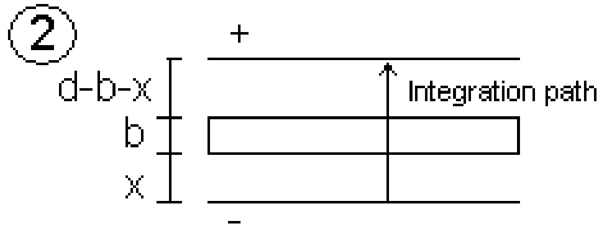
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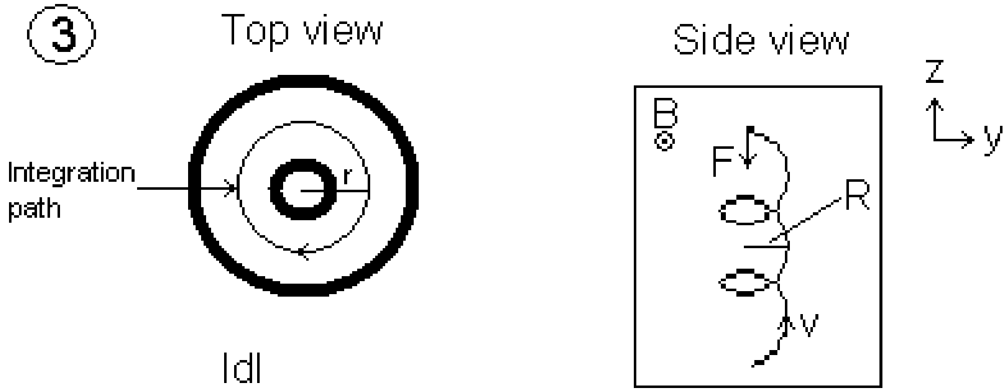
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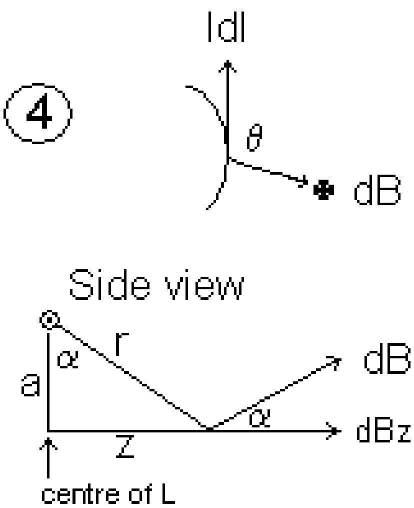
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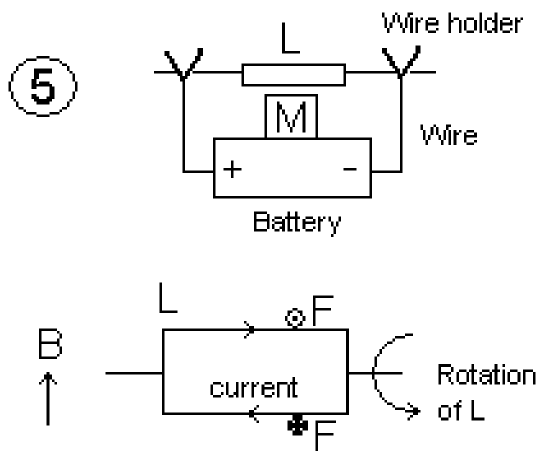
centre of L


5
L
Wire holder
M
Wire
+
-
Battery
L
F
B
current
F
Rotation of L