2002-2003 Physics Olympiad Preparation Program - University of Toronto - Solution Set 6: Circuits and Modern Physics

1(a) At $t = 0$, the capacitor is not charged. The voltage across the capacitor, $V(t)$, is zero. For $t > 0$, charges build up on the capacitor's plates. One plate becomes positively and the other negatively charged; $V(t)$ increases, but it cannot do so indefinitely. Only a finite number of charges can be added or removed from the plates. The voltage smoothly approaches V, the voltage of the battery.
1(b) As in part (a), $V(t)$ starts at zero and increases with time. However, when $V(t) = V' < V$, $V(t)$ sharply drops to zero. This corresponds to the spark between the plates. The capacitor then begins recharging and $V(t)$ increases as before until $V(t) = V'$, after which $V(t)$ falls to zero again. The graph has the appearance of a sawtooth waveform.
1(c) The peak voltage or period can be changed by altering the resistance and/or capacitance. For example, R determines how quickly $V(t)$ approaches V'. If it takes a long time for $V(t)$ to reach V', the period of the sawtooth waveform will be longer. The same is true if the capacitor is nearly an ideal device with $V' \approx V$. If the capacitor is very poor, then $V' \ll V$ and the sparking happens more often. The period of the waveform will be smaller.

2(i) The periodic functions $cos(\omega t - kz)$ and $sin(\omega t - kz)$ advance in the positive z-direction. The electromagnetic wave propagates in the z-direction with a phase velocity $v_{phase} = \frac{\omega}{k} = c$, where c is the speed of light. (ii) Suppose the wave consists of quantized energy packets known as photons. According to quantum theory, each photon has an energy $\epsilon = pc$ and a momentum $p = \frac{\epsilon}{c}$. The electromagnetic wave therefore has a linear momentum $P_z = \frac{E}{c}$ since it propagates along $\hat{z}$. (The total energy of the wave is $E = IAT$, where I is the wave intensity, A is an area, and T is a time.) (iii) If the wave interacts with a particle that has a charge $q > 0$, mass m, and is confined to the xy plane ($z = 0$), then the particle's motion is described by $\mathbf{F} = m\mathbf{a} = q\mathbf{E}$ or

$$ma_x = qE_x \rightarrow m\frac{d^2x}{dt^2} = qE_0 cos(\omega t) \; , \tag{1}$$

$$ma_y = qE_y \rightarrow m\frac{d^2y}{dt^2} = qE_0 sin(\omega t) \; . \tag{2}$$

Verify by differentiation with respect to t that the following

$$x(t) = -\frac{qE_0}{m\omega^2}cos(\omega t) \; , \quad v_x(t) = \frac{qE_0}{m\omega}sin(\omega t) \; , \tag{3}$$

$$y(t) = -\frac{qE_0}{m\omega^2}sin(\omega t) \; , \quad v_y(t) = -\frac{qE_0}{m\omega}cos(\omega t) \; , \tag{4}$$

satisfy the above equations. The particle acquires an angular momentum

$$\mathbf{L} = \mathbf{r}(t) \times \mathbf{p}(t) = \frac{q^2E_0^2}{m\omega^3}\hat{z} \; , \tag{5}$$

about the z-axis, where $\mathbf{r}(t) = x(t)\hat{x} + y(t)\hat{y}$ and $\mathbf{p}(t) = m(v_x(t)\hat{x} + v_y(t)\hat{y})$ are the particle's position and momentum. The electromagnetic wave therefore has an angular momentum about the z-axis.

3(a) An observer who is at rest with respect to the mirrors will "see" the light pulse travel straight up and down. The time required for a round trip is $t_0 = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c}$, where c is the speed of light.

3(b) An observer who is travelling along $\hat{y}$ at speed V relative to the mirrors will "see" the clock travel along $-\hat{y}$ and the light pulse move at an angle to the vertical. The speed of light c is the same for all observers. The time t required for a round trip is

$$2\sqrt{\left(\frac{Vt}{2}\right)^2 + L^2} \quad = \quad ct \rightarrow t = \frac{2L}{\sqrt{c^2 - V^2}} = \frac{\frac{2L}{c}}{\sqrt{1 - V^2/c^2}} = \frac{t_0}{\sqrt{1 - V^2/c^2}} \; . \qquad (6)$$

As the observer's speed V approaches c, the time it takes for the light pulse to return to its origin approaches infinity. Since material objects such as an observer cannot move at the speed of light, the light pulse is always seen to return to its starting point, although the time t could be very large.

4(a) An electron (e) orbiting a proton (p) at a radius r_0 experiences electrostatic and centripetal forces. To find the kinetic energy K of the electron set

$$F_E \quad = \quad F_c \rightarrow \frac{1}{4\pi\epsilon_0}\frac{|q_e q_p|}{r_0^2} = m_e\frac{v^2}{r_0} \rightarrow K = \frac{1}{2}m_e v^2 = \frac{1}{8\pi\epsilon_0}\frac{|q_e q_p|}{r_0} \; . \qquad (7)$$

The potential energy U and total energy E_{tot} of the electron are

$$U \quad = \quad -\int_{\infty}^{r_0} F_E dr = -\int_{\infty}^{r_0} \frac{1}{4\pi\epsilon_0}\frac{q_e q_p}{r^2}dr = -\frac{1}{4\pi\epsilon_0}q_e q_p\left(-\frac{1}{r}\right)_{\infty}^{r_0} = \frac{1}{4\pi\epsilon_0}\frac{q_e q_p}{r_0} \; , \qquad (8)$$

$$E_{tot} \quad = \quad K + U = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r_0} \; , \qquad (9)$$

where $q_e = -e$ and $q_p = +e$. The binding energy is $E_b = -E_{tot} \approx 13.6166972$ eV where $e = 1.60217733 \times 10^{-19}$ C and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m.
4(b) The mass of the hydrogen atom is less than the sum of the electron and proton masses when they are measured separately. This mass difference is equivalent to the atom's binding energy according to Einstein's mass-energy relation,

$$E_b \quad = \quad -\Delta E = -\Delta mc^2 = -(m_H - m_e - m_p)c^2 = 13.87926537\text{eV} \; , \qquad (10)$$

where $c = 2.99 \times 10^8$ m/s. The two calculations are in fairly good agreement. This is surprising since the electron in a hydrogen atom cannot be viewed as a classical particle orbiting the proton. The measurements of the masses and fundamental constants have to be very accurate in order for this comparison to be reasonable.

5. Try several measurements using different lengths of wire L between X and Y. Use the same type of wire in each case in order to keep the resistivity ρ and the cross section A of the material between X and Y constant. The light bulb shines less brightly as L is increased. Less current flows through the circuit for a given battery voltage V. The resistance R of the wire increases with L. If a voltage V is applied to a conducting object, then a current I and a current density $j = \frac{I}{A}$ are generated. The resistivity of the object is defined to be

$$\rho \quad \equiv \quad \frac{E}{j} = \frac{V/L}{I/A} = \frac{V/I}{L/A} = \frac{R}{L/A} \rightarrow R = \frac{\rho L}{A} \; , \qquad (11)$$

where $E = \frac{V}{L}$ is the applied electric field and R is the resistance of the object. This demonstrates the relationship between R and L, if the currents and potentials are not too large.

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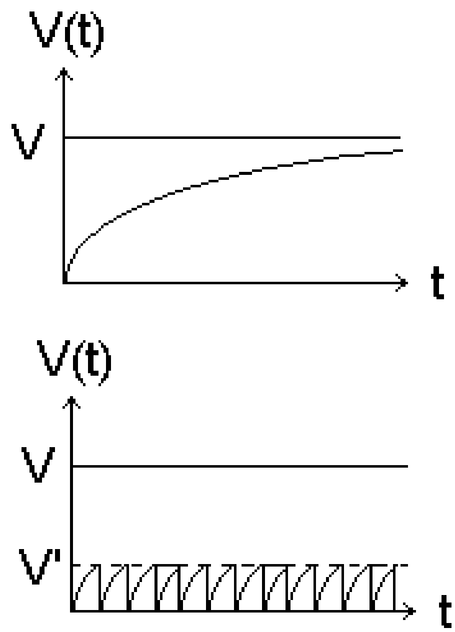

3