

2003-2004 Physics Olympiad Preparation Program

– University of Toronto – Solutions. Set 1: General

1. Lily and Ann in Wonderland.

Two girl friends spent almost all their summer vacation in the Water Park. Lily liked to slip down the plane slide LW with small angle of inclination, while Ann liked the arc slide AW. Their start points L and A were at the same height. Both slides were cut at the water level, where arc had its tangent line coincided with the horizontal direction. The girls always started down simultaneously. Try to calculate who first achieved the water in the pool. You may neglect friction.

Solution

To solve a problem we always replace the everyday life terminology with math and physics notions and definitions. In our case taking into account the real proportions of the objects' sizes we can assume the girls to be point masses.

Thus, our problem now has a following statement:

Two identical point masses start from a rest simultaneously slipping down along the frictionless surfaces. One mass slides along the inclined plane LW with small angle of inclination, another one – along the arc AW. Their starting points L and A are at the same height. Horizontal line is tangent to the arc at the final point W.

Compare the times of slide along two tracks to the point W.

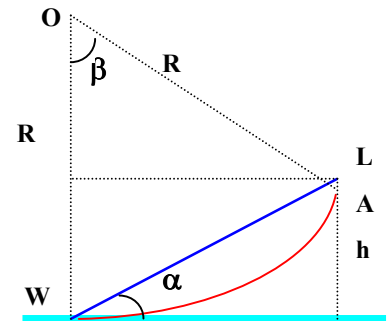


Fig.1

Let us add some imaginary details to our figure: **O** – center of curvature of the arc AW; **R** – its radius; **h** – the height of starting points L and A; **alpha** - angle of inclination of the plane LW.

According to the law of conservation of energy the speed of two masses at the point W is the same:

$$v_f = \sqrt{2gh}$$

Sliding along the incline LW is a uniformly accelerated motion with acceleration vector component parallel to velocity $g_v = g \sin \alpha$ (1), and g – acceleration due to gravity. The elapsed time for the motion along plane (or line LW) is:

$$\Delta t_p = \frac{v_f}{g_v} = \frac{1}{\sin \alpha} \sqrt{\frac{2h}{g}} \quad (2)$$

Sliding along the arc is not uniformly accelerated motion. The tangential instantaneous acceleration, resulting in the speed growth at any point of the arc, may be calculated according to Eq.(1). Acceleration is decreasing with the decrease of the slope along the arc.

Intent look at the problem text permits us apply a model of a simple pendulum to the motion of point mass along the arc. The words “small angle” in physics or mathematics always mean that we can assume that

$$\alpha \approx \sin(\alpha) \approx \tan(\alpha). \quad (3)$$

Attention! In this and similar formulas angle has units of *radians*!

Let us convert the terms of our problem into the terms of the problem of simple pendulum. It is possible because of the same trajectory of the point mass. Thus, our radius of curvature for the arc can be considered the length of the string of simple pendulum; unknown elapsed time $\Delta t_a = T/4$; T is the period of pendulum oscillations; β is the amplitude of oscillations, and $\beta = 2\alpha$. And the well-known equation for this period gives the solution of our problem

$$\Delta t_a = \frac{1}{4} \times 2\pi \sqrt{\frac{R}{g}} \quad (4)$$

From the Fig.1 it is obvious that $R = R \cos(\beta) + h = R \cos(2\alpha) + h$;

$$R[1 - \cos(2\alpha)] = h$$

$$R = \frac{h}{2 \sin^2(\alpha)}$$

Substituting R in Eq.4 gives us the solution for Δt_a :

$$\Delta t_a = \frac{\pi}{4 \sin(\alpha)} \sqrt{\frac{2h}{g}} \quad (5)$$

Thus, for the ratio of elapsed times we obtain $\frac{\Delta t_a}{\Delta t_p} = \frac{\pi}{4} < 1$

In spite of the longer distance covered, the travelling time down the arc appears to be less than the elapsed time for the motion along the inclined plane.

The above statement was proved **only** for small angles α and β , because we used the solution for the simple pendulum with small amplitude (angle β). This matches our problem, if $\alpha < 10^\circ$, and when we are quite satisfied with the error of about 2%. If we need the error of less than 1%, angle of inclination should be limited by 5° .

At this point we could stop and assume the problem to be solved.

However, for anyone who is interested not only in marks there appears a question.

What is the general solution for any angle α ?

To answer this question we may use two different ways. First of all, it is important to understand that the angle of inclination α cannot exceed 45° in our problem, and $\beta_{\max} = 90^\circ$.

a) We may again apply the model of pendulum to our point mass on the arc track. Now this approach expects the knowledge of formula for the period of non-harmonic oscillations of the pendulum with large amplitude (try to find it in the serious handbooks on physics):

$$T = 2\pi \sqrt{\frac{l}{g}} \left[1 + \left(\frac{1}{2}\right)^2 \sin^2 \frac{\beta}{2} + \left(\frac{1}{2} \times \frac{3}{4}\right)^2 \sin^4 \frac{\beta}{2} + \left(\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6}\right)^2 \sin^6 \frac{\beta}{2} + \dots \right] \quad (6)$$

In brackets we see a series with diminishing terms. When a simple pendulum has small amplitude $\beta \leq 10^\circ$ even the second term is negligible (1.9×10^{-3}) in comparison with 1, which is the first term in the series. All other contributions are much less. That is why you are always

investigating only the small amplitude oscillations of pendulum with its period of $T = 2\pi\sqrt{\frac{l}{g}}$.

And now you know that this solution is approximate.

Let us estimate the ratio of elapsed times for the largest possible angle of our problem

$\alpha = 45^\circ = \beta/2$. Calculation for the terms of Eq.6 results in following:

n	1	2	3	4	5
t_n	1	0.125	0.0350	0.0120	4.67×10^{-3}

The fifth term contributes less than 1% in the sum of series, and we can calculate the sum of five terms in brackets as 1.18. For this case the ratio of times is equal to:

$$\frac{\Delta t_a}{\Delta t_p} = \frac{\pi}{4} \times 1.18 = 0.927 < 1 \quad (7)$$

We have obtained qualitatively the same result even for the large angle of inclination.

This may be explained in words as a result of greater acceleration on the arc track at the first half of distance.

b) Another way to compare the times is connected with application of derivatives and integrals. Any new position on the arc trajectory corresponds to the new value of instantaneous velocity. Let us choose so little time interval between two observations of the sliding point mass that the velocity during this interval can be considered a constant. To accent the tiny little value of time interval and the corresponding distance we will put a symbol of differential “d” before these and other very small quantities. In this case we can define the instantaneous velocity as $v = \frac{dl}{dt}$ (8).

The total elapsed time will be a sum of very small

time intervals $dt = \frac{dl}{v}$ (9), obtained from Eq.8.

The symbol of such **sum** is \int - elongated letter **S**, and is called an **integral**. Now we may

calculate the elapsed time as $\Delta t_a = \int \frac{dl}{v}$ (10). To calculate integral to the right of Eq.10, it is

necessary to substitute two variables dl and v with combination of constants, given quantities and only one variable. Parameters h and α must be known. In our problem and in the majority of other problems on the circular motion the best choice for variable is angle φ . Then $dl = -R \times d\varphi$, (we put minus because while l is increasing, φ decreases, or in other words: the positive change in distance dl corresponds to the negative change in angle $-d\varphi$),

$$R = \frac{h}{2\sin^2 \alpha}, \quad v = \sqrt{2gz}, \quad z = R \cos(\varphi) - R \cos(2\alpha) = \frac{h}{2\sin^2(\alpha)} [\cos(\varphi) - \cos(2\alpha)]$$

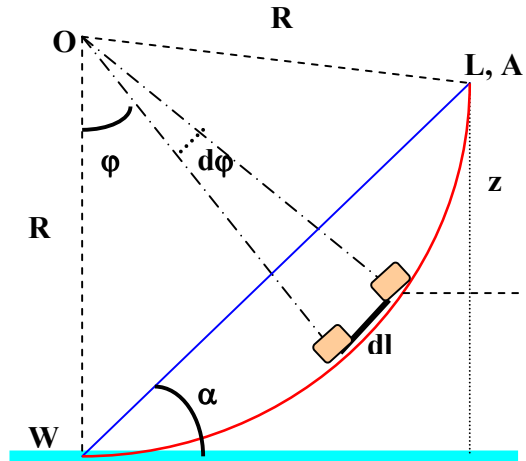


Fig.2

Let us rewrite Eq.10 after all substitutions with new variable:

$$\Delta t_a = \frac{1}{2\sin(\alpha)} \sqrt{\frac{h}{g}} \int_{2\alpha}^0 \frac{-d\varphi}{\sqrt{\cos(\varphi) - \cos(2\alpha)}} \quad (11)$$

The little symbols over and beneath the integral symbol mean domain for our variable, which changes during the motion from 2α to 0 .

Integral in equation (11) can be calculated only numerically with needed precision. You may try to do it with the help of computer programs MathCAD or Maple for different angles α .

Our problem limits the value of inclination by $\alpha = 45^\circ = \pi/4$.

For this angle elapsed time along the arc is

$$\Delta t_a = \sqrt{\frac{h}{2g}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{\cos(\varphi)}} \quad (12)$$

Application of the above mentioned program gives for the integral value of 2.57.

The ratio of elapsed times (if you haven't yet forgot the question of our problem) results in

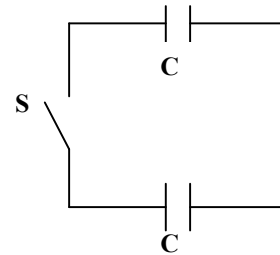
$$\frac{\Delta t_a}{\Delta t_p} = 0.9 < 1$$

We can conclude that the time elapsed on the arc trajectory is always less than the time on its chord for any angle of inclination in the range of $0 \leq \alpha \leq \frac{\pi}{4}$.

2. Let us disprove the Law of Conservation of Energy!

Two identical capacitors with capacitance C are connected with each other and with switch S as shown. One of the capacitors initially had charge Q . Another one was uncharged. What was the electric field energy initially? After switch S is closed and system comes to equilibrium, calculate the total energy of the electric field of the system. Where does the energy "disappear"?

Examine two cases: a) significant and b) negligible resistance of the wires in the circuit.



Solution

The initial energy of the system is $E_{ci} = \frac{Q^2}{2C}$. This is the energy stored in the charged capacitor.

If our system were ideal in any case of the wires' resistance value, we would never wait until it comes to electrostatic equilibrium. The equilibrium will occur in infinite time period. This is a result of exponential law for charging and discharging of the ideal capacitor through the ideal wires.

Nevertheless, our common sense and everyday experience tells us that in reality we **can** observe the equilibrium in some finite time interval after we have closed the switch. At the attained electrostatic equilibrium the total energy stored in two identical charged capacitors

with total charge Q is equal to $E_{ct} = 2 \times \frac{(Q/2)^2}{2C} = \frac{Q^2}{4C}$, which is a half of the initial energy.

The task is to enumerate all the processes when the system of two capacitors is missing the energy.

At some specific conditions and constructive peculiarities of the switch the first process of energy loss is **a spark**. This process is accompanied by ionization of the surrounding medium molecules, heating of the environment, and radiation in a wide range of electromagnetic wavelengths.

At the same time, in vacuum a spark cannot take place, and we should seek other ways of energy leakage. For both cases a) and b) we can point out two of them: *heat* and *radiation*.

Law of conservation of energy gives us $E_{ct} = E_h + E_r$. The most amazing thing in this

result is the independence of the sum $E_h + E_r = \frac{Q^2}{4C}$ on the value of wires' resistance!

It means that physical properties of wires influence the distribution of energy between portions E_h and E_r .

a) Non-zero resistance of a wire always entails its heating. The power delivered to a conductor of resistance R is often referred to as **I^2R loss**. In our problem this expression represents the instantaneous value because in theory the current is decreasing according to the exponential law. Moreover, in real wires resistivity always depends upon the temperature. For initially significant resistivity we can neglect its variation with temperature because of a comparatively little initial current according to the Ohm's law, and consequently low temperature of the wire during all process.

Thus the power is a product of two time dependent variables: **R and I** . Besides, they influence each other.

Any current is a source of magnetic field. When current is not steady, magnetic field around the wire is not constant. Variable magnetic field generates the variable electric field and vice versa. Actually this process is a radiation of electromagnetic waves. In our problem the waves are not well known sinusoidal waves. Their frequency will decrease in time as well as the amplitude. The same can be described from another point of view. The ordered motion of electrons in the wire in our problem is the motion with deceleration because the current

decreases from the initial value $I_i = \frac{Q}{CR}$ to zero.

Radiation of electromagnetic waves always accompanies the slowing down charged particles. The greater resistivity of the wire is, the greater the ratio E_h / E_r .

b) An ideal, or "perfect", conductor would have zero resistivity. Today we know a great number of metals and compounds with zero resistivity below a certain critical temperature, T_C . The phenomenon of zero resistivity is called **superconductivity**.

In absence of resistance we must take into account another property of the wires – their inductance L . The value of inductance depends upon the material, length, and cross-section area of the wire. Hence zero resistance makes our circuit to operate as a so-called **LC circuit** with two connected in series identical capacitors. The frequency of charge, current, and

potential difference oscillations in our circuit is $f = \frac{1}{2\pi} \sqrt{\frac{2}{LC}} = \frac{1}{\pi\sqrt{2LC}}$

Such circuit is a source of electromagnetic waves and acts as a broadcasting antenna.

Thus in the case b) all missed energy transfers into the energy of radiation.

3. Why not to use heat when it is hot?

Heat engine is a device that takes in heat from a high-temperature source or heat reservoir, converts part of this heat into work, and releases the remaining heat to the environment at a lower temperature or to the low temperature reservoir. The gasoline engine in a car is also a heat engine. The productiveness of any heat engine can be defined by calculating its efficiency. For this problem you need to find as much information as you can about the specific heat engine that was designed by Carnot and its efficiency.

Now suppose that you wish to increase the efficiency of a Carnot engine as much as possible. You can do this by increasing T_H (temperature of the high-temperature reservoir) a certain amount while keeping T_L (temperature of the low-temperature reservoir) constant. Or by decreasing T_L the same amount while keeping T_H constant. Which would you do? Prove your choice quantitatively.

Under what conditions would a Carnot engine be 100% efficient?

Imagine you have all facilities and funds to realize any scientific idea. What object should you choose as the best low-temperature reservoir for your heat engine? Of course, you are very smart and thrifty.

Solution

The efficiency η_i of an ideal heat engine described in 1824 by a French engineer Sadi Carnot has the highest theoretical value among other possible heat engines. Besides, this quantity does not depend on the properties of working substance, but only on the temperatures of low-

temperature and hot-temperature reservoirs: $\eta_i = 1 - \frac{T_L}{T_H}$. You can find a lot about the Carnot

engine on the web site www.phy.ntnu.edu.tw/java/carnot/carnot.html.

Now we should find out which change by the same amount ΔT gives the greater rise of efficiency: the increase of the hot temperature or the decrease of the low temperature. In the

first case we can obtain efficiency $\eta_{i+} = 1 - \frac{T_L}{T_H + \Delta T}$; in the second case: $\eta_{i-} = 1 - \frac{T_L - \Delta T}{T_H}$.

We can examine the result of the temperature changes in both cases by comparing their difference with zero or by comparing their ratio with 1.

$$\eta_{i+} - \eta_{i-} = -\frac{T_L}{T_H + \Delta T} + \frac{T_L - \Delta T}{T_H} = \frac{-T_L T_H + T_L T_H + T_L \Delta T - T_H \Delta T - \Delta T^2}{T_H (T_H + \Delta T)} = \frac{-(\Delta T (T_H - T_L + \Delta T))}{T_H (T_H + \Delta T)}$$

Denominator of this fraction is positive, and numerator is negative, thus, the difference of efficiencies is negative. It means that the decreasing of temperature in the low-reservoir temperature by ΔT results in greater efficiency than the increasing by the same value of temperature in the high-temperature reservoir. The same conclusion can be obtained from the

calculation of the ratio $\frac{\eta_{i+}}{\eta_{i-}}$. This ratio is smaller than 1.

The Carnot engine is an ideal heat engine with only reversible processes and without any energy loss other than the amount released to the low-temperature reservoir. That is why we can only imagine the 100% efficiency for this ideal engine. 100% efficiency means $\eta_i = 1$.

This occurs when the temperature in the low-temperature reservoir is 0 K (absolute zero). In practice we can only approach this temperature but never reach.

What can be chosen as the best low-temperature reservoir? First of all, it must be a reservoir with a constant temperature, and our expenses for the maintenance of this temperature must be as small as possible. That is why we should choose any nature reservoir of huge volume and with as low temperature as possible.

The best one on the Earth is the great oceans, especially its deeper levels. If you are on the space station you will choose the outer space with approximately 2.7 K temperature and infinite volume.

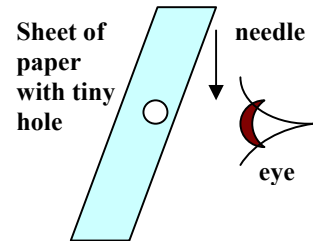
4. Inverted image without lens and mirror.

Take a standard sheet of paper and pierce it with a sharp needle.

a) Try to look through this tiny hole and start your observation keeping the sheet at the approximate distance of 20 cm. Then little by little advance the hole in the direction of one of your eyes until the paper touches your eyelashes. Describe what you see looking at and through this hole during this approaching and explain the result.

b) Keeping the sheet at the approximate distance of 1" from your eye and a needle vertically above the hole and at the middle distance between your eye and the sheet, slowly lower the needlepoint attentively looking at the hole behind it. To protect your eye put on any glasses. It may be either transparent sunglasses or glasses with lenses. Lenses do not play any role in this experiment except eye protection.

Describe the results of observation of the needlepoint motion against the background of the hole and explain them.



Solution

This experiment results in the observations that have many similarities and a number of distinctions in comparison with the observation of an image in a pinhole camera. You were familiarized with this device in the Physics 11 textbook. The best way to arrange our experiment is to hold a sheet of paper with a pinhole between your eye and illuminated window. To specify the conditions of experiment let us consider a diameter of the eye pupil equals to 2 mm, and the diameter of a pinhole to be four times smaller.

a) In the first part of experiment you begin with the arm length distance between your eye and a pinhole. At this distance you see nothing beyond the sheet of paper, since a pinhole is too little as well as the pupil of your eye (Fig.1). Very small aperture angle ϑ_1 permits to obtain almost no information about the objects above our sheet.

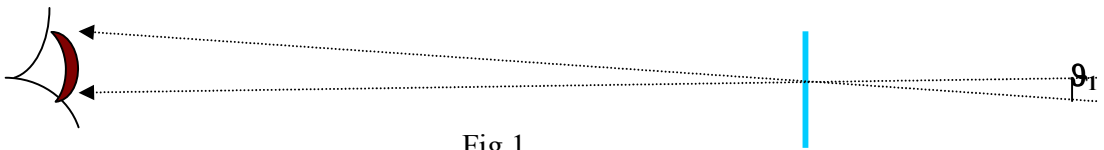


Fig.1

Approaching the pinhole to the eye we obtain the greater aperture angle ϑ_2 and begin to see large objects above the hole (Fig.2).

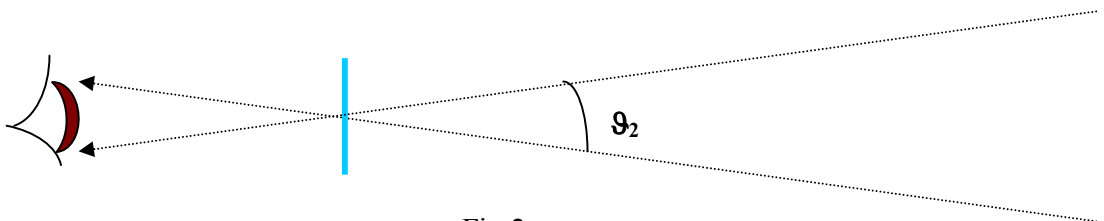


Fig.2

Up to this point the results of our experiment coincide with the same in a pinhole camera. The difference is in the attitude of the image. Through a pinhole we see an upright image of all objects above. On the contrary, in a pinhole camera the image on the screen is inverted. From the sketches in Fig.1 and Fig.2 it cannot be understood. We need more detailed picture.

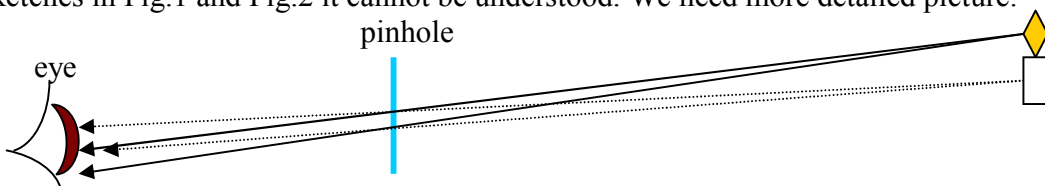


Fig.3

Since a pinhole has a finite diameter, our eye can see a part of the same rays from the same segments of object, as it took place before the sheet of paper with a pinhole hidden some of them. There is no distortion of the rays, and the image is upright, less illuminated and clearer. The last is also a result of a smaller number of rays coming into the eye.

b) If you have done the experiment with the needle or pin thoroughly, you could observe the inverted image of the needle edge beyond the hole.

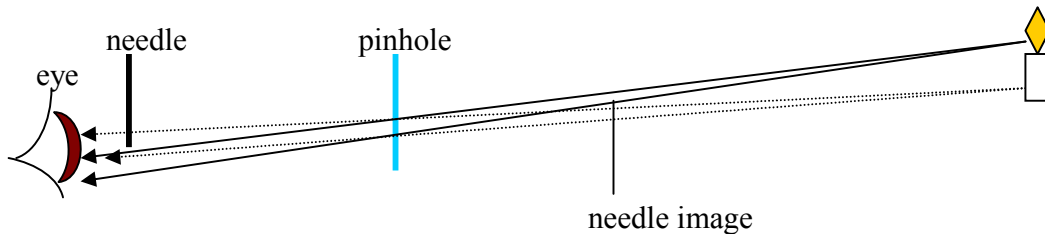


Fig.4

Fig.4 helps to understand this phenomenon. When the needle is moving downwards close to your eye, it first cuts off the rays from the bottom of an object. On the other hand, the needle is too close to the eye to be seen. You see exactly the same result, as it would be seen if a needle was situated beyond the hole and was moving upwards. That is why it seems to you that you see an inverted image of a needle. It looks more like a shade of the needle than virtual image. Both parts of this experiment show the importance of honest recording of the experimental results even when they were unexpected or disproved our expectations!

5. She is so young and pretty!

To solve this problem you should remember the general decay law from your high school Mathematics. The radioactive decay of nuclei is usually described quantitatively with the parameter T . Let us define the half-life, T , as the time in which $1/2$ of the particles decay.

The natural uranium ore now consists of $\eta_1 = 99.28\%$ ^{238}U and $\eta_2 = 0.72\%$ ^{235}U . Half-life periods of ^{238}U and ^{235}U nuclei are correspondingly equal to $T_1 = 4.47 \cdot 10^9$ years and $T_2 = 0.70 \cdot 10^9$ years. Estimate the Earth's age assuming the amounts of two isotopes were equal at the moment of birth of our planet. Find in any handbook on Physics the corresponding data, and calculate the percentage difference between the tabled data and your result.

Solution

The radioactive decay law may be expressed in different ways. The plainest formula for it is:

$$N = N_0 \times 2^{-\frac{t}{T}}$$

N_0 – number of nuclei of some isotope at the moment of $t=0$;

N – number of nuclei that did not decay through the time interval t ;

T – half-life time of the isotope. When $t = T$, $N = N_0 / 2$.

According to our problem we have the same values of N_0 for both isotopes. Assuming that the age of the Earth equals the time of decay t , we can calculate the present-day number of nuclei of both isotopes:

$$N_1 = N_0 \times 2^{-\frac{t}{T_1}} \quad (1)$$

$$N_2 = N_0 \times 2^{-\frac{t}{T_2}} \quad (2)$$

At the same time

$$N_1 = \eta_1 \times (N_1 + N_2) \quad (1')$$

$$N_2 = \eta_2 \times (N_1 + N_2) \quad (2')$$

First, we should equate (1) to (1'), and (2) to (2'):

$$N_0 \times 2^{\frac{t}{T_1}} = \eta_1 \times (N_1 + N_2) \quad (3)$$

$$N_0 \times 2^{\frac{t}{T_2}} = \eta_2 \times (N_1 + N_2) \quad (4)$$

This is a system with two unknowns: t and N_0 . Left and right parts of both equations have the same factors. In such case division of (3) by (4) gives the shortest solution. Division results in

$$2^{t(\frac{1}{T_2} - \frac{1}{T_1})} = \frac{\eta_1}{\eta_2}$$

At this point we cannot advance without knowledge of what is **logarithm**. Upon the definition "logarithm to the base b of x " $\log_b x$ is the exponent to which the base b must be raised to give

x . In our example the base is 2, $x = \frac{\eta_1}{\eta_2}$. Thus, we can write that

$$t(\frac{1}{T_2} - \frac{1}{T_1}) = \log_2(\frac{\eta_1}{\eta_2})$$

Now it is possible to write the solution for unknown age of the Earth t :

$$t = \frac{T_1 T_2 \log_2(\frac{\eta_1}{\eta_2})}{T_1 - T_2}$$

Calculations give $t = 5.90 \times 10^9$ years.

On the Internet site

<http://www.talkorigins.org/faqs/faq-age-of-earth.html>

you can find the latest dating of the Earth's age, the comparison of the dating methods, and find out what initial considerations make our result so different from the newest data (4.9×10^9 years).

The percentage difference between the tabled and calculated in our problem time is 20.4%.