2003-2004 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 2: Mechanics

1. **Up or Down?**

Two masses $\mathbf{m_1}$ and $\mathbf{m_2}$ are connected by an ideal spring with coefficient of elasticity **k** (Fig.1). Initially this system is in equilibrium.

a) What is the direction and minimum magnitude of displacement of the upper mass from the position of equilibrium to make lower mass to jump up?

b) How does the answer to the question (a) depend on the ratio $\mathbf{m_1/m_2}$? Explain the result.

Solution

The problem evidently should be solved using the law of conservation of energy. First of all, it is necessary to define zero potential energy. This must be made in the most convenient way, because actually nobody calculates the perfect value of this kind of energy but only a difference in potential energies of different states of the system. In the problem we deal with the gravitational potential energy.

One of such convenient 0-levels can coincide with the top plane of block 2 (Fig.1). In this case we will need two additional quantities that are not given: $\mathbf{L_0}$ - the natural length of the coil spring, and **d** – the thickness of block 1. We will not find them in the resultant equation.

As it was not told otherwise, we assume the spring obeys the Hooke's law always in our problem. It depends not only on the coefficient of elasticity, or stiffness coefficient, **k**, but also on the masses of the blocks.

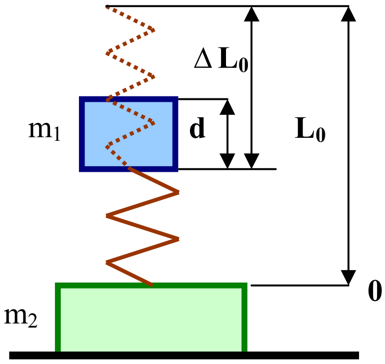

Fig.1

In equilibrium position the spring is compressed by the value of $\Delta \mathbf{L_0}$. To apply the Hooke's law we must be sure that $\Delta L_0 << L_0$. If it is not so, we cannot solve the problem in general. For the little value of compression the Hooke's law gives us

$$\Delta L_0 = \frac{m_1 g}{k} \qquad (1)$$

Let us examine two possible directions of the block 1 displacement and find two corresponding values ΔL_{up} and ΔL_{down}, counted off from the equilibrium position.

1) ***Displacement upward.***

The process is following: we are slowly lifting the block 1 upwards until the block 2 does not touch any more the surface of a table.

During this process there are three forces experienced by the block 2:

force of gravity $F_{g2} = m_2 g$ (interaction with the Earth);

normal force N_2 (interaction with the table);
spring force F_{S2} (interaction with the coil spring).
Since block 2 is in equilibrium, we can write the conditions for it as a vector equation:

$$\vec{F}_{g2}+\vec{N}+\vec{F}_{S2}=0$$

At the moment the block 2 starts upwards it does not experience normal force any more. For this moment we have $\vec{F}_{g2}=-\vec{F}_{S2}$. It can be rewritten for the magnitudes of forces as

$$m_2 g = kx_{up} \qquad (2)$$

where x_{up} is the excess of the length of the spring over its natural length.
The magnitude of displacement from the position of equilibrium appears to be:

$$\Delta L_{up} = \Delta L_0 + x_{up} = \frac{m_1 g}{k}+\frac{m_2 g}{k}=\frac{g}{k}(m_1+m_2) \qquad (3)$$

This equation is symmetric relative to **m₁** and **m₂**. We cannot even guess which mass was initially the upper one and which one was at the bottom.

2) Displacement downward.

Block 2 is lowered by the unknown value of ΔL_{down} from the position of equilibrium and then is released. The greatest change in the natural length of the spring is now $x_{down}=\Delta L_0+\Delta L_{down}$ (4).
To calculate the initial potential energy of the system let us exclude the value of potential energy of the block 2. This value will not change during the process and thus it is out of interest.
At the lowest position of the block 1 potential energy of the system (without block 2) is

$$E_i = m_1 g\left(L_0 - x_{down}+\frac{d}{2}\right)+\frac{k(x_{down})^2}{2} \qquad (5)$$

After spring is released, the block 1 rushes upwards. When its center of mass reaches the height of

$$L_0 + x_{up} + \frac{d}{2},$$

traveling with speed v, the block 2 starts upwards. We obtain the same equation (2) for x_{up}

$$m_2 g = kx_{up}$$

The total energy of the system at this moment is

$$E_f = m_1 g\left(L_0 + x_{up}+\frac{d}{2}\right)+\frac{m_1 v^2}{2}+\frac{k(x_{up})^2}{2} \qquad (6)$$

After equating expressions (5) and (6) we should substitute x_{up} and x_{down} by their values from Eqs.(2) and (4), and simplify result.
All above gives us the following value for ΔL_{down}

$$\Delta L_{down}=\sqrt{\left[\frac{g}{k}(m_1+m_2)\right]^2+\frac{m_1 v^2}{k}}$$

The question of our problem is to calculate the least displacement $\Delta L_{down\,\min}$. It evidently matches the minimal value of root and takes place when $v=0$. The result is

$$\Delta L_{down}=\frac{g}{k}(m_1+m_2). \qquad (7)$$

The comparison of Eqs. (30 and (7) gives us: $\Delta L_{up}=\Delta L_{down}$.

So, we see that both the direction and magnitude of the minimal displacement relatively to the position of equilibrium do not depend on the ratio of masses. We can turn our system upside-down and obtain the same answer. Here must be mentioned that the position of equilibrium will change in this case.

Now we should explain why the solution does depend only upon the sum of masses but not upon their relative initial position.

Fig.2 shows the potential energy of the system as a function of the spring length ***x.*** The function is a parabola. It was obtained from Eq.5, substituting $L_0 - x_{down} = x$. You can derive the vertex form of parabola from this expression yourself and find that the position of vertex has coordinates $\left(L_0 - \Delta L_0, E_{pot\,\min}\right)$, $E_{pot\,\min} = k\Delta L_0\left(L_0 - \frac{\Delta L_0}{2} + \frac{d}{2}\right)$. We have exactly the position of equilibrium when $x = L_0 - \Delta L_0$. At the position of stable equilibrium the potential energy function of any system always has its minimum. The final result of the first block displacement is the same for any direction of displacement.

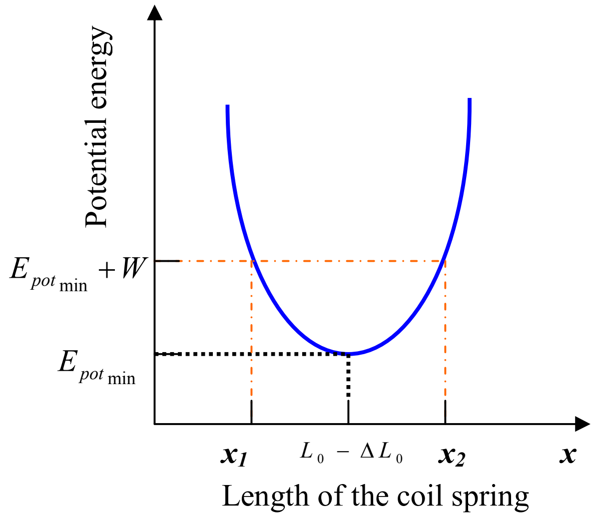

Fig.2

It is extension of the spring by the value x_{up} and lifting of the center of mass to the same height. Thus, we can say that we produced some work ***W*** to change the potential energy of the system by one the same amount either by stretching or by compressing the spring. Due to the symmetry of the potential energy function this result is obtained either by positive or by negative change in the length of spring from its equilibrium position length:

$$\frac{x_1 + x_2}{2} = L_0 - \Delta L_0$$

In our solution we assumed the absence of any missing energy for heating or irreversible change of the spring's dimensions.

2. Winter Physics

Two connected sledges with masses $\mathbf{m_1}$ and $\mathbf{m_2}$ are sliding down the snow hillside with the angle of inclination α (Fig.2). Coefficients of kinetic friction between sledges and snow are μ_1 (new sledges) and μ_2 (old sledges), and $\mu_1 < \mu_2$

Calculate the acceleration of the center of mass for two different kinds of connection:

a) with a string; b) with a rod.

Explain the result.

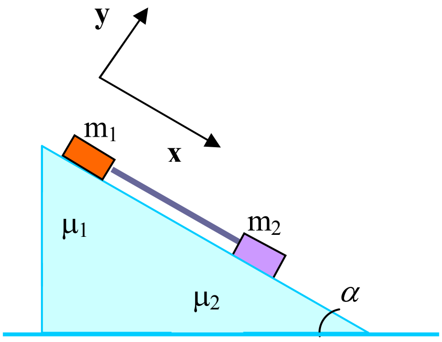

Fig.3

Solution

Acceleration of the center of mass of two blocks cannot depend upon the internal forces of the system, such as the forces between a rod and each block and the force experienced by each block from another one after they begin to move as a one object in case (a).
We have two external forces experienced by the system of two blocks: force of gravity and force of friction. Both forces do not depend on the position or velocity of the blocks. Thus, we can write that the center of mass has acceleration down the hillside, as follows:

$$a_{cx} = \frac{(m_1 + m_2)g\sin\alpha - \mu_1 m_1 g\cos\alpha - \mu_2 m_2 g\cos\alpha}{m_1 + m_2} = g\left(\sin\alpha - \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2}\cos\alpha\right) \quad (1)$$

If it seams too easy and dubious we can examine the solution by calculating the acceleration step by step for two different kinds of tie.

a) Connection with a string.

Acceleration of blocks according to the second Newton's law is equal to following:

$$a_{1x} = g\sin(\alpha) - \mu_1 g\cos(\alpha) \qquad (2a)$$

$$a_{2x} = g\sin(\alpha) - \mu_2 g\cos(\alpha) \qquad (2b)$$

Acceleration of the center of mass can be derived from the expression for the position of the center of mass. Upon the definition, it is given as

$$\vec{r} = \frac{\Sigma\, \vec{r}_i \times m_i}{\Sigma m_i}$$ (Σ - the sum of terms, *i* – the current number of the term in a set)

The corresponding displacement of the center of mass is equal to

$$\Delta\vec{r} = \frac{\Sigma\, \Delta\vec{r}_i \times m_i}{\Sigma m_i} \qquad (3)$$

From Eqs.2 we see that the motion of two blocks is the straight-line uniformly accelerated one. For such motion displacement is directly proportional to the acceleration. Thus, for the last we can write

$$a_x = \frac{\Sigma\, a_{ix} \times m_i}{\Sigma m_i} = \frac{a_{1x} m_1 + a_{2x} m_2}{m_1 + m_2} = g\sin(\alpha) - \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2} g\cos(\alpha) \qquad (4)$$

Eqs.(1) and (4) coincide!
According to the statement of problem $\mu_1 < \mu_2$. That is why the first block will overtake in some time the lower block, and they will start to move with one the same acceleration.
The second and the third Newton's laws give for each block:

$$\begin{aligned} m_1 a_x &= m_1 g\sin(\alpha) - \mu_1 m_1 \cos(\alpha) - R \\ m_2 a_x &= m_2 g\sin(\alpha) - \mu_2 m_2 \cos(\alpha) + R \end{aligned} \qquad (5)$$

Each block experiences the force R from another one.
After addition of these equations and division both parts by the sum of masses we will again obtain the result of Eq.(1).

b) Connection with a rod.

If the rod is considered to be weightless and perfectly incompressible, we can apply equations (5) and their consequences to our system, with the force R experienced by each block from the rod. Expression for $\boldsymbol{a_x}$ remains the same.

3. **Are You Sure You Can Weigh?**
An equal-arm balance with a uniform weigh beam of mass *M*, thickness *d*, and length *2L* is in horizontal equilibrium position on the platform, which is moving with uniform acceleration *g/2*. Two equidimensional cubes with side *b* are motionless relative to the weigh beam (Fig.3).
Calculate the ratio of densities of two cubes.

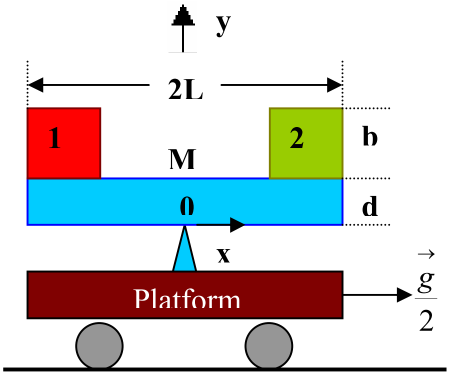

Fig.3

Solution

The problem may be solved by a number of ways. Let us observe the simplest one.

First of all, the dimensional analysis hints that we cannot solve the problem for the unknown ratio of densities if the only one mass *M* is given and plays a significant role in the solution. We simply could not compose the dimensionless expression, including only one mass.

That is why we should investigate 1) under what conditions the solution exists without additional information, and 2) what is not given but we need to know to solve the problem in general.

For the system "Two cubes + beam" forces of friction are internal forces. An equilibrium condition for the center of mass of the system in accelerating (non-inertial) frame of reference will not include these forces.

Let us find a position for the center of mass. Our Cartesian system has its origin at the pivot point **0**. Upon the definition the position of the center of mass is equal to:

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + M\vec{r}_3}{m_1 + m_2 + M} \qquad (1),$$

where $\vec{r}_i$ - are the position vectors of the centers of mass of two blocks and a beam. Always when we have a vector equation we need to rewrite it for the vectors' components. This gives:

$$y_c = \frac{m_1y_1 + m_2y_2 + My_3}{m_1 + m_2 + M} = \frac{m_1\left(d + \frac{b}{2}\right) + m_2\left(d + \frac{b}{2}\right) + M\frac{d}{2}}{m_1 + m_2 + M} \qquad (2y)$$

$$x_c = \frac{-m_1\left(L - \frac{b}{2}\right) + m_2\left(L - \frac{b}{2}\right)}{m_1 + m_2 + M} = \frac{(m_2 - m_1)\left(L - \frac{b}{2}\right)}{m_1 + m_2 + M} \qquad (2x)$$

We can see that $y_c > 0$ always, and $x_c > 0$ for $m_2 > m_1$.

In accelerated reference system net torque is 0 because in this frame of reference beam is in equilibrium and both blocks are motionless. Inertial force has an opposite direction to the direction of horizontal acceleration of the system. Let us equate the torques:

$$(m_1 + m_2 + M)\frac{g}{2}y_c = (m_1 + m_2 + M)gx_c$$

or

$$y_c = 2x_c \qquad (3)$$

Eqs.(2x) and (3) mean that $x_c > 0$, $m_2 > m_1$, and $\rho_2 > \rho_1$.

After a short simplification we obtain:

$$(m_2 + m_1)\left(\frac{d}{2} + \frac{b}{4}\right) + \frac{Md}{4} = (m_2 - m_1)\left(L - \frac{b}{2}\right) \tag{4}$$

If $M << m_2 + m_1$, and $d << b$ then we can neglect the second term in the left side of Eq.(4), and solve the problem completely. Thus, for the negligibly light and negligibly thin beam

$$(m_2 + m_1)\frac{b}{4} = (m_2 - m_1)\left(L - \frac{b}{2}\right)$$

As the volumes of two blocks are equal, we obtain the following:

$$\frac{\rho_1}{\rho_2} = \frac{L - \frac{3}{4}b}{L - \frac{b}{4}} < 1 \tag{5}$$

When we have to take into account both the mass and the thickness of a beam, we need to know one more quantity, e.g. *m₂*. Suppose, we have included this value into our set of given parameters. It entails the following equation:

$$m_2\left(\frac{\rho_1}{\rho_2} + 1\right)\left(\frac{d}{2} + \frac{b}{4}\right) + \frac{Md}{4} = m_2\left(1 - \frac{\rho_1}{\rho_2}\right)\left(L - \frac{b}{2}\right)$$

$$\frac{\rho_1}{\rho_2} = \frac{L - \frac{3}{4}b - \left(\frac{M}{m_2} + 2\right)\frac{d}{4}}{L - \frac{b}{4} + \frac{d}{2}} < 1 \tag{6}$$

The ratio of densities from Eq.(6) is smaller than the ratio from Eq.(5).
Here it is necessary to mention restrictions on M/m_2 and d that have appeared in Eq.(6).
As the numerator must be positive, we can express these restrictions in a pair of inequalities:

$$\frac{m_2}{M} > \frac{d}{4L - 3b - 2d} \tag{7M}$$

$$d < 2L - \frac{3}{2}b \tag{7d}$$

If values of M and d do not match conditions (7), the beam becomes the main part of the system, and will turn counterclockwise.

4. A Science Fiction Story

On December 8, 2103, Nick, Natalie and Paul were arguing about the possibility of traveling with a speed exceeding the speed of light in vacuum. Paul was sure such speed was impossible for any object. Nick and Natalie thought that the speed might be any if the proper frame of reference was chosen. They decided to prove that the Einstein's theory had already become outdated. They took their spacecrafts and very soon reached the velocities of the opposite directions and of **0.8c** each (**c**=3x10^8 m/s), respectively Paul, who remained on the Earth.

What was the speed of Natalie's spacecraft, measured by Nick?

Solution

The motion of two spacecrafts occurs in one dimension, say along x-axis in opposite directions. We should choose an appropriate formula from the Lorentz velocity transformation. Let the Nick's spacecraft have the velocity $v_{x1} = +0.8c$ with respect to the Earth, and Natalie's one have the velocity $v_{x2} = -0.8c$ in the same frame of reference. According to the Lorentz formula the velocity v'_{x2} of Natalie's rocket in the Nick's frame of reference will be

$$v'_{x2} = \frac{v_{x2} - v_{x1}}{1 - \frac{v_{x1} v_{x2}}{c^2}} \tag{1}$$

$$v'_{x2} = \frac{-0.8c - 0.8c}{1 - \frac{(-0.8c)0.8c}{c^2}} = -0.98c$$

It is clear that if Natalie decided to measure the Nick's velocity, she would obtain +0.98c.
Both of them can increase the speed of the spacecraft but will never reach the value of relative speed that exceeds ***c.***
However, there is one inaccuracy in the above solution. It concerns the rule about the number of significant digits in the result. In our problem the resultant value must contain only one significant digit and must be written as 1c. That is why in the problems on special relativity you can find usually the velocity values rounded to such number of significant digits that permits you to write the result, which would not disprove Einstein's theory.
If we do not change the given values of two speeds, e.g. to the values of 0.800c with the result of 0.980c, Nick and Natalie can say that according to the accuracy of the experiment their relative velocity is equal to the speed of light!

5. The Bouncing Ball (experiment).

Take any ball that won't break when hits the floor. You will probably need a ruler and a timer. Your task is to explore the kind of collisions of the bouncing ball with a floor.
You know two simple kinds of collisions: *elastic* and *perfectly inelastic* ones. Real collisions that we can observe in our everyday life can be only approximately elastic.

a) Create preferably a dimensionless parameter to describe any collision quantitatively, as a degree of approximation to elastic or perfectly inelastic collision. Calculate the range of this parameter.
b) Perform an experiment with the bouncing ball to find out the value of the created parameter in the collisions of the ball with the floor at your room. Compose a chart with your measurements and primary calculations.
c) Calculate the proposed parameter for your specific experiment. Calculate the errors of measurements. Make a conclusion on the kind of the observed collisions: were they more like elastic or perfectly inelastic collisions.
d) Why does the ball of any kind never reach one the same height more than once?

Solution.

a) It was mentioned above that we know two extreme cases of collisions: elastic and perfectly inelastic ones.
In the first type of collisions there is no loss of kinetic energy, i.e. the total kinetic energy of isolated system before collision is equal to the value of total kinetic energy after collision.
In the perfectly inelastic collisions the speed of two colliding objects after collision is the same. In other words, the objects stick together and move as a one rigid body after collision. A portion of kinetic energy of the system is transformed to internal energy of the objects deformed during collision.

The linear momentum is conserved in both types of collisions.
We can distinguish the collisions of different elasticity by one of two dimensionless parameters:

$$k_1 = \frac{E_{kf}}{E_{ki}} \quad \text{or} \quad k_2 = \frac{|\Delta E_k|}{E_{ki}} \qquad (1)$$

where E_{ki} – kinetic energy of the system before collision; E_{kf} – kinetic energy after collision; $\Delta E_k = E_{kf} - E_{ki}$. Evidently $k_2 = 1 - k_1$; $0 \le k_1 < 1$; $0 < k_2 \le 1$. $k_1 = 0$, and $k_2 = 1$ are the extreme values of the parameters for perfectly inelastic head-on collision of two objects with equal masses. After the relationship between k_1 and k_2 is clarified, it is quite enough to investigate one of them, e.g. k_1. This value is easier to be obtained from the direct measurements.

b) At the starting point of the ball its potential energy is equal to its total energy, and the potential energy of the floor is not expected to change during the experiment. Thus, we can state that the total energy of our system E_{total} is equal to the potential energy of the ball of mass *m* at its highest point with the altitude h_0:

$$E_{total} = mgh_0$$

Just before a hit a ball has its kinetic energy $E_{ki} = E_{total}$. Just after the collision its kinetic energy is equal to $E_{kf} = mgh_1$, where h_1 – is the altitude of a ball after its first bounce. Actually this value is not a total energy of the system, because as a result of collision the floor, building and even the Earth have got a fraction of kinetic energy E_{ki} of the ball. Through the kinetic energy is energy of motion, and there is no evident motion of the complex object “floor+building+theEarth” after collision, we can assume the kinetic energy of a ball to be a new value of total kinetic energy of the system. To calculate coefficient k_1 we should measure h_0 and h_1, and substitute the ratio of kinetic energies in Eqs.(1) by the ratio of altitudes:

$$k_1 = \frac{h_1}{h_0} \qquad (2)$$

To make the result representative we should repeat the altitude measurements after as many bounces, as we can, say *N*. The results must be put into a chart, e.g. as following:

The number of a bounce, *i*	h_i, *cm*	k_i	Δk_i
0	h_0		
1	h_1	$k_1 = h_1/ h_0$	$\|k_1 - k_{av}\|$
2	h_2	$k_2 = h_2/ h_1$	$\|k_2 - k_{av}\|$
3	h_3	$k_3 = h_3/ h_2$	$\|k_3 - k_{av}\|$
4	h_4	$k_4 = h_4/ h_3$	$\|k_4 - k_{av}\|$
……..	……	…..	…..
N	h_N	$k_N = h_N/ h_{N-1}$	$\|k_N - k_{av}\|$

c) After calculations of the values k_i in the third column of our chart we should not hurry into averaging them. To be sure that the values k_i can be averaged we should plot them in the *“k_i vs. i"* Cartesian coordinate system. If it becomes obvious that all of the plots are dispersed along a straight line with zero slope, we can calculate the average value k_{av} as:

$$k_{av} = \frac{\Sigma k_i}{N}$$

Then the forth column should be filled, and the resultant statistical error of measurements can be calculated as:

$$\Delta k_{av} = \frac{\Sigma \Delta k_i}{N} \qquad (3)$$

Taking into account that measurements of values h_i were very difficult and thus inaccurate we must consider the value 10% and less of the average error to be quite acceptable.

Result must be recorded in the form of:

$$k_1 = k_{av} \pm \Delta k_{av} \qquad (4)$$

In the above-examined case coefficient k_1 does not depend upon the speed of a ball, and depends only on the properties of two objects: a ball material and the floor-cover material.

If Eq.(4) gives $0 \le k_1 < 0.5$, the collisions are more like inelastic ones. Out of this range k_1 indicates the collisions more like elastic ones.

If, on the other hand, the plots show some different behaviour of $k_i(i)$, or $k_1(v)$, function, we cannot average the obtained set of values, and can only represent the plots as results of our measurements.

d) A ball never reaches one the same height more than once because during each collision a portion of its potential energy is transformed into the kinetic energy of the floor and building vibrations, internal energy of a ball, and internal energy of the floor.