2003-2004 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 4: Optics

1. Lenses can be not only round.

The external diameter of a thin glass tube is much greater than the diameter of the internal capillary. The index of refraction of the glass is 4/3. Through the side surface the diameter of the capillary seems to be equal to *d=2.66 mm.*
What is the real diameter of the capillary in the tube?

Solution

In this problem we see a virtual image formed by the rays refracted at the cylindrical surface of the glass tube. Thus, we can consider the tube to be a cylindrical lens. Our solution needs a detailed drawing.
In Fig.1 the tube cross-section is shown.

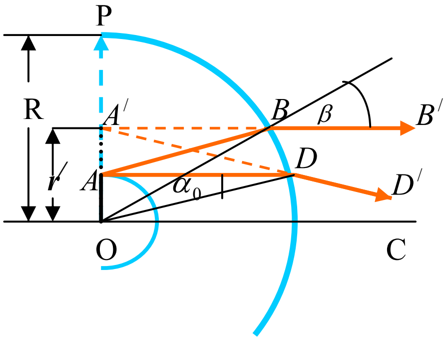

Fig.1

R is the exterior radius of the tube; $r^/ = d/2 = OA^/$, is the apparent radius of the capillary; ***OA = r*** is the real radius of the capillary; ***OC*** is the principal axis of our cylindrical lens. Refracted rays $ABB^/$ and $ADD^/$ form the image of the apparent point $A^/$. We could choose any two rays from point *A* to locate its image (upon our convenience). In our solution lines *AD* and $A^/B^/$ are parallel to the axis *OC*.
In general, point $A^/$ lies to the left from the vertical line *OP*, which coincides with radius. You can determine the position of point $A^/$ by constructing ray diagrams for different ratios of ***r/R*** thoroughly applying the Snell's law for our given value of the index of refraction.
If ***R*** significantly exceeds ***r,*** say more than ten times, point $A^/$ appears on the OP. Let us specify the relationship between ***R*** and ***r*** as ***r/R < 0.1*** (*). Besides, we need to introduce the angles $\alpha = \angle ABO$ (is not shown), α_0, and β; and somehow symbolize the unknown diameter of the capillary as, for instance, ***y (y=2r)***. We have following system of equations:

$$n\sin(\alpha) = \sin(\beta) \quad \text{(the Snell's law)} \qquad (1)$$

$$R = \frac{r^/}{\sin(\beta)} \quad \text{(triangle } A^/BO\text{)} \qquad (2)$$

$$\frac{r}{\sin(\alpha)} = \frac{R}{\sin\left[\frac{\pi}{2} + (\beta - \alpha)\right]} \quad \text{(sine law for triangle } ABO\text{)} \qquad (3)$$

The last equation can be changed to:

$$\frac{r}{\sin(\alpha)} = \frac{R}{\cos(\beta - \alpha)} \qquad (4)$$

After substitution *R* from eq.(2) we obtain the following:

$$r = \frac{r^{/}}{n\cos(\beta - \alpha)} \qquad (5)$$

To estimate the value of denominator let us consider all angles in *radian* measure. It is obvious that any angle *ABO* is less than α_0. Upon our assumption $\sin(\alpha_0) \leq 0.1$ (see condition(*)). That is why $\sin(\alpha\ ) \leq 0.1$. From the Snell's law $\sin(\beta) \leq \frac{2}{15}$. The little value of the both angles permits to replace the sine functions by the angles in radians, as $\sin(\alpha) \approx \alpha$, and $\sin(\beta) \approx \beta$. The difference appears only in the third decimal. After all simplifications in eq.(5) we obtain

$$r = \frac{r^{/}}{n\left[\sqrt{1-\alpha^2}\sqrt{1-\beta^2} + \alpha\beta\right]} \approx \frac{r^{/}}{n} \qquad (6)$$

Unknown diameter

$$y = \frac{d}{n} = \frac{2.66x3}{4} \approx \frac{8}{3}x\frac{3}{4} = 2mm$$

2. A crooked ray

The index of refraction ***n*** of the planet X atmosphere decreases with the altitude ***h*** as $n = n_0 - \alpha h$, where α is a constant. The planet radius equals ***R***.

At what altitude the ray of the light can propagate around the planet along the circumference?

<u>Solution</u>

In the atmosphere, where the index of refraction changes with altitude, the light can propagate not along the straight line. Upon the definition, *n= c/v,* where $c = 3 \cdot 10^8 m/s$. The speed of light ***v*** increases with the decrease of ***n,*** when altitude rises. If the wave front is not horizontal, it turns; light rays refract permanently and propagate along the curve.

Let us choose two rays that form a thin light canal of width Δh at the altitude of h_0, as it is shown in Fig.2.

To observe "a ray" propagating along circumference we must assume that $\Delta h << h_0$ (1).

One of the rays at the altitude h_0 turns around the full circumference in the time interval

$$t = \frac{2\pi(R + h_0)}{v_1} = 2\pi(R + h_0)\frac{n_0 - \alpha \cdot h_0}{c}$$

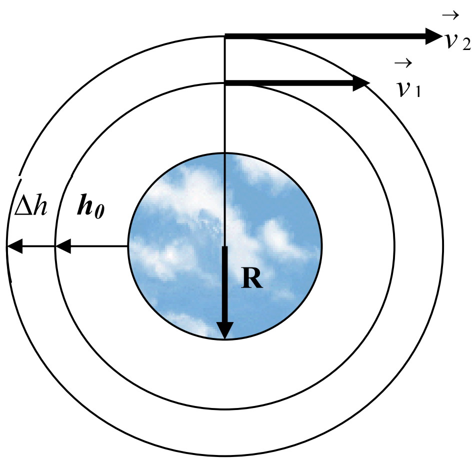

Fig.2

The propagation along the circumference with the radius $R + h_0 + \Delta h$ will take the second ray, forming our light canal, the same time. This is the requisite condition to have a wave front always perpendicular to the curve of propagation. That is why, for the second ray we have

$$t = \frac{2\pi(R + h_0 + \Delta h)}{v_2} = 2\pi(R + h_0 + \Delta h)\frac{n_0 - \alpha \cdot (h_0 + \Delta h)}{c}$$

Equating the times for two rays, and taking into account the relationship (1), we obtain that

$$h_0 = \frac{1}{2}\left(\frac{n_0}{\alpha} - R\right)$$

The described phenomenon is called a circular refraction, and theoretically can be observed, for example, in the atmosphere of Venus.

3. Mirrors in wave optics.

A parallel beam of quasi-monochromatic light with wavelength $\lambda = 500nm$ falls on the system of two plane mirrors 1 and 2 with the angle of incidence $\alpha = 30^0$ (Fig.3). Partly the beam is reflected from the surface of the semi-transparent mirror 1, partly – by the stationary mirror 2. Reflected rays pass through the lens *L* to the detector *D* in the focal plane of the lens. Signal in the detector is proportional to the intensity of the received wave. What is the frequency of the signal in the detector if mirror 1 is moving upward with the speed $u = 0.01 cm/s$, as it is shown in Fig.3?

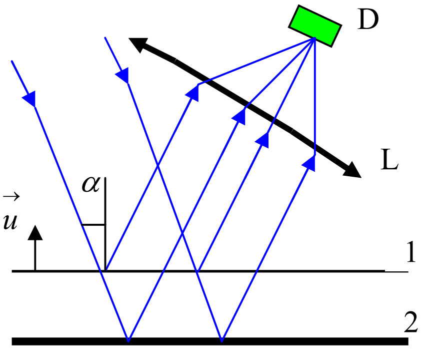

Fig.3

Solution

Let us examine the time moment when the position of mirror 1 relative to the position of mirror 2 is given by the coordinate *z* in Fig.4. At this moment the path difference between the rays *BDL* and *ACL* is given by:

$$\Delta = AC + CD - BD \qquad (1)$$

Fig.4 shows that

$$AC = CD = \frac{z}{\cos(\alpha)} \qquad (2)$$

$$BD = AD\sin(\alpha) = 2z\tan(\alpha)\sin(\alpha) = \frac{2z\sin^2(\alpha)}{\cos(\alpha)} \qquad (3)$$

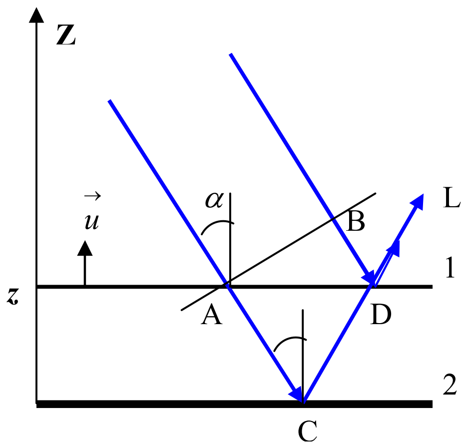

Fig.4

After substitution of *AC, CD*, and *BD* in eq.(1) from eqs. (2) and (3) the path difference becomes as follows:

$$\Delta = \frac{2z}{\cos(\alpha)} - \frac{2z\sin^2(\alpha)}{\cos(\alpha)} = 2z\cos(\alpha)$$

Detector registers the constructive interference, when $2z\cos(\alpha) = m\lambda$, where *m = 0, 1, 2, ...*

In the time interval between two consecutive detected peaks of intensity mirror 1 covers a distance of $\delta z = \frac{\lambda}{2\cos(\alpha)}$. The mentioned time interval equals $T = \frac{\delta z}{u} = \frac{\lambda}{2u\cos(\alpha)}$.

The frequency of the signal is

$$f = \frac{1}{T} = \frac{2u\cos(\alpha)}{\lambda} = 346 Hz$$

4. Transparent barrier.

Light is an electromagnetic wave and can be described as a propagation of alternating electric and magnetic fields that create each other. Vectors of electric field ***E*** and magnetic field ***B*** in the wave are always perpendicular to each other and to the direction of propagation. Waves in which the electric and magnetic fields are restricted to be parallel to certain directions are said to be linearly polarized waves. As sunlight is a mixture of a diversity of waves the, sunlight beam is not polarized wave. There exist crystals and organic films named polarizers that can transmit through them only the waves with specific direction of electric field called the polarizing direction. Waves of the beam with other directions of the field vectors pass through such materials only partly depending on the relative direction of the electric field vector of the wave and the polarizing direction of the material. If the angle between the polarizing direction of the polarizer and the electric field vector of the wave is θ, and the intensity of the incident linearly polarized beam I_0, we can calculate the intensity of the exit beam after passing through the polarizer according to the Malus's law:

$$I = I_0 \cos^2(\theta)$$

The sunlight is unpolarized. Its intensity is reduced by the factor of one half after passing a single polarizer, as the average value of $\cos^2(\theta)$ is ½.

We also must remember that the polarizers usually are not ideal, and can absorb the light as all other transparent materials.

In our problem we deal with two polarizers and the beam of sunlight. The transmitted intensity after the first polarizer reaches $\eta_1 = 30\%$ of the incident sunlight beam intensity. In some arrangement of two identical polarizers, the transmitted intensity will decrease to $\eta_2 = 13.5\%$ of the incident sunlight beam.

Calculate the angle between the polarizing directions of these two polarizers in the described experiment.

Solution

Two processes cause attenuation of the intensity of the transmitted light.

The first one is imperfect transmission of the waves with electric vector ***E*** that does not coincide with the specific polarizing direction in the polarizer pointed by the transmission axis. This kind of the reducing of the incident beam intensity is expressed in the Malus's law:

$I = \frac{1}{2} I_0$ $\quad (1^{/})$ for unpolarized primary beam,

$I = I_0 \cos^2(\theta)$ $\quad (1^{//})$ for polarized primary beam with the angle θ between vector $\boldsymbol{E_0}$ and the transmission axis of the polarizer. Laser light is always linearly polarized.

The second source of attenuation is the absorption of the electromagnetic energy of the wave in the material of the polarizer due to the interaction of the material electrons with the wave. Such kind of intensity reduction is observed in all materials, even those we consider to be transparent.

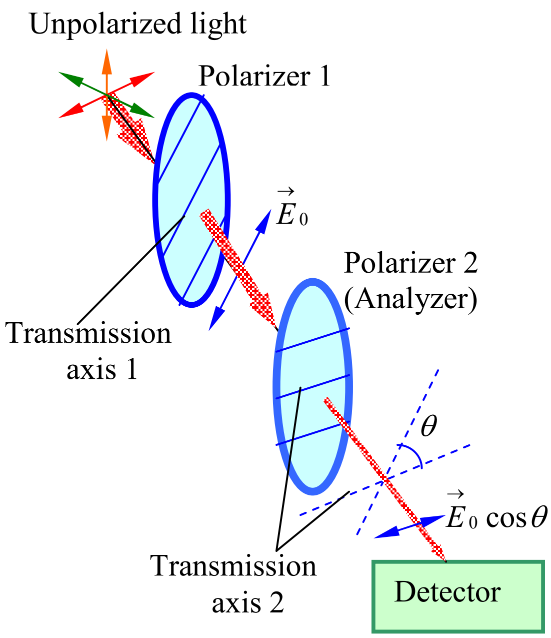

Fig.5

To specify the fraction of the absorbed intensity we can use the coefficient of absorption R or the coefficient of transmission τ, which is not connected with polarization process.

Fig.5 shows, how unpolarized light (sunlight) passes through a pair of polarizers. Let us symbolize the intensity of sunlight beam as I_S. After the first polarizer the intensity of the exit beam is

$$I_0 = \frac{I_S}{2}\tau \qquad (2)$$

After the second polarizer the intensity of the exit beam is given by eq.($1^{//}$):

$$I = I_0 \cos^2(\theta)\tau \qquad (3)$$

Substituting I_0 from eq.(2), we can obtain the following:

$$I = \tau^2 \frac{I_S}{2}\cos^2(\theta) \qquad (4)$$

In the statement of our problem it is given that

$$I_0 = \eta_1 I_S \quad (5^{/}), \text{ and } I = \eta_2 I_S \quad (5^{//})$$

From eqs.(2) and ($5^{/}$) we can calculate the coefficient of transmission

$$\tau = 2\eta_1 \qquad (6)$$

From eqs.(4) and ($5^{//}$) we can express the cosine of θ through the τ:

$$\cos^2(\theta) = \frac{2\eta_2}{\tau^2} = \frac{2\eta_2}{4\eta^2{}_1}$$

$$\cos(\theta) = \sqrt{\frac{\eta_2}{2\eta_1{}^2}} = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6} = 30^O$$

5. **Not only music.**

Explain in details the optical phenomena that can be observed with the help of ordinary compact disk. Describe as many results of observation as you can.

Solution

CD is one of the most amazing objects to observe.

First of all, we should mention that it demonstrates simultaneously the phenomena of the ray optics (geometrical optics), with its plastic coverage working as a plane mirror, and the wave optics through the beautiful results of dispersion of sunlight.

This remarkable optical phenomenon in the case of CD can be understood as a result of diffraction of the incident light on the circular grating, intense reflection of particular wavelength at certain angles and further constructive interference. Thus, we can enumerate optical phenomena that can be observed with the help of CD: reflection, diffraction, dispersion of sunlight, and interference.

Compact disks contain a large number of concentric circular tracks on their surface and the spacing between these tracks is very small (about few microns). That is why, when light from a source like the sun or a light bulb strikes the series of the tracks, they work as a reflection diffraction grating.

Since compact disk has circular tracks, not straight gratings, the picture created by diffraction does not look like series of spectra with straight and parallel spectral lines (regions with the same color), but like series of concentric circular spectra.

Unfortunately, we can observe the full circles very rarely in the specific case when the incident light beam is directed along the axis of rotation of CD. Usually we can see only sectors of the spectra or arcs (see Fig.6). According to the theory of diffraction the angle of diffraction increases with wavelength.

That is why CD produces a spectrum in the order of violet, closest to the origin direction, to red.

The majority of results depend on the arrangement of the experimental devices: position of the source of light, angle between the beam of light and CD surface, position of the observer.

Sometimes, if the angle of incidence of the rays to the surface of the disk is high, we can even see spectra in the form of lines, which go from the center of the disk to the perimeter (Fig.7). The interference patterns appeared on CD may be used to detect, study, and measure light.

For example, if we know the distance *d* between the tracks on the CD surface, then we can use it to measure the wavelengths of light emitted by any source using the grating. Conversely, we can use a known wavelength of a particular color of light to measure the tiny distance between tracks on a CD.

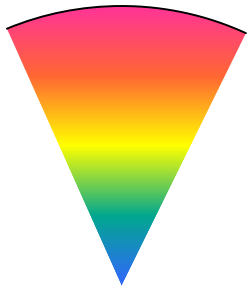

Fig.6

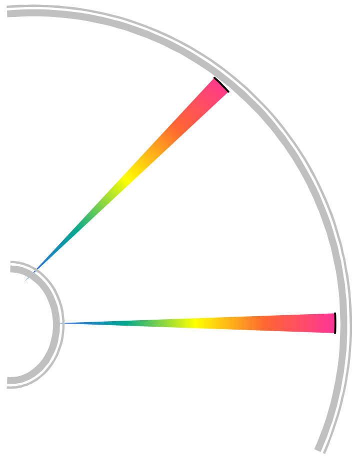

Fig.7