2003-2004 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 5: Electricity and Magnetism

1. Always-fidgety molecules.

We can imagine a polar molecule, as a system with two point masses carrying unlike charges with invariable distance between them (Fig.1).
This molecule is put in the uniform electric field with the strength *E=300 V/cm*. The molecular average length is $l = 10^{-8}$ *cm*, the masses of charges are $m = 10^{-24}$ *g*, the magnitude of charges is equal to $e = 1.6x10^{-19}$ *C*.
Find the period of oscillations of the molecule in the field.

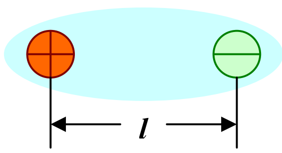

Fig.1

Solution

In the position of stable equilibrium a polarized molecule, or a dipole, is always oriented along the lines of electric field (Fig.1-1).
If this balance is slightly disturbed, there appears a torque of the electric force couple $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ that is shown in Fig.1-2. This torque rotates dipole around its center of mass and constrains the molecule to return to the equilibrium position.
If we consider the charges separately, forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ act as restoring forces for two simple pendulums with the length *l/2*. This similarity permits to write a period of oscillations for each charge as follows:

$$T = 2\pi\sqrt{\frac{l}{2a}} \qquad (1)$$

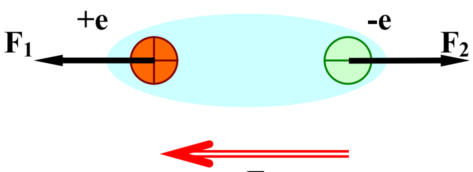

Fig.1-1

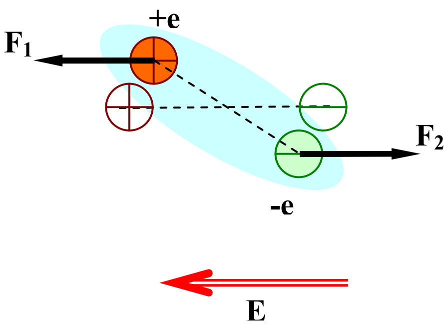

Fig.1-2

a is acceleration of each charge in the electric field.
According to the second Newton's law

$$\vec{a} = \frac{\vec{F}}{m} \qquad (2)$$

In the electric field $\vec{F}_1 = e\vec{E}$, $\vec{F}_2 = -e\vec{E}$ (3).
After substitution of the magnitude of acceleration in eq.(1) by the solutions of eqs.(2) and (3) one can obtain the following period of oscillations:

$$T = 2\pi\sqrt{\frac{lm}{2eE}} \approx 2 \cdot 10^{-11} s$$

2. Lightning production.

In the high voltage electrostatic generator electric charges are carried by the belt, made of a special insulating material, and charge a large metallic sphere with radius $R=1.5\ m$, as it is shown in Fig.2. Width of the belt is $l=1m$, its speed is 20 m/s. A discharge in the air takes place when the strength of electric field exceeds $E=30\ kV/cm$.

Estimate the maximal values of voltage and current that can be obtained with such generator.

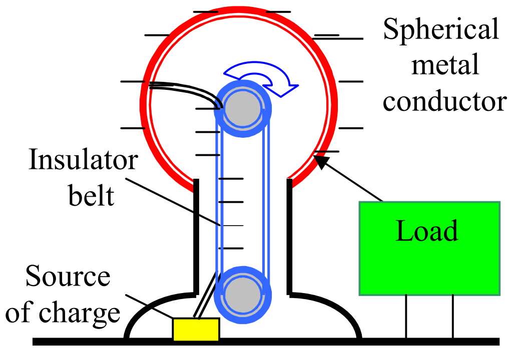

Fig.2

Solution

The shown generator is named a Van de Graaff generator. This is a huge device that occupies a room with height of several storeys. It is used to obtain a very significant voltage.

A maximal voltage between the Earth and a charged sphere in the generator depends upon the air conductivity. If electric field by the sphere surface exceeds the given value of $E = 30\ kV/cm$, charge leaks off in sparks between the sphere and some nearest grounded points. The same phenomenon in the atmosphere we observe between a cloud and the Earth during a storm.

The field around the sphere is neither uniform, nor symmetric because a sphere is not a sole conductor in the space. Thus it is difficult to calculate accurately a voltage or potential distribution around the sphere. We can only estimate voltage between the sphere and the Earth.

First supposition must determine how to consider the electric field of the Earth. We know that our planet is charged negatively, is surrounded by the electric field with $E_S \approx -130V/m = -1.30V/cm$ on its surface, and has non-zero potential.

It can be easily shown that we should not take into account the electric field of the Earth.

Electric field is a vector field. Let us draw a vector of the resultant field just by the sphere surface. Suppose we have chosen a point A (Fig.2-1) very close to the sphere. The resultant vector of electric field $\vec{E}_{res}$ at this point has almost the same magnitude and direction as in the absence of the Earth due to the value of $E/E_S \approx 23000$. For any point by the sphere surface we can come to the same conclusion.

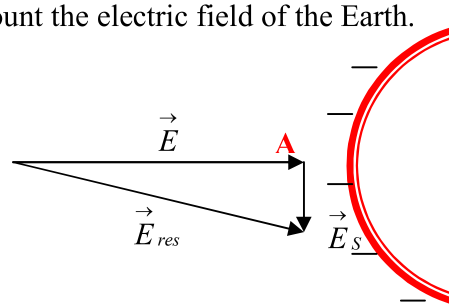

Fig.2-1

In particularly, it means that for our estimation the field around the sphere can be calculated as a field of a solitary sphere in the space. If we had only a sole charged sphere in the empty space, its field E_1 and potential φ_1 would be related as for the field of a point charge, i.e.:

$$E_1 = \frac{\varphi_1}{R} \qquad (1)$$

According to eq.(1) the maximal potential of the charged sphere in generator is equal to:

$$\varphi_{\max} = E \cdot R \approx -4.5 \cdot 10^6 V \qquad (2)$$

Eq.(2) estimates also the maximal voltage as we neglected other charged objects in surroundings.

The maximal current I_{max} is the maximal rate of charge deposition from the belt on the sphere. It is also determined by the air conductivity. Electric breakdown between the charged belt and the Earth can occur before the sphere is charged critically.

To estimate the maximal current we can assume the belt to be an infinite plane with its surface charge density σ. From the relationship between E, ε_0 (electric permittivity), and σ for a plane the maximal value of charge density $\sigma_{\max}$ is following:

$$\sigma_{\max} = 2\varepsilon_0 E \qquad (3)$$

Using eq.(3) it is easy to calculate the maximal current:

$$I_{\max} = \sigma_{\max} lv = 2\varepsilon_0 Elv \approx 10^{-3} A \qquad (4)$$

The calculated results clarify why the generator is usually very big.

3. Electronics.

The diode is connected in the circuit with given electro-motive force ε, as it is shown in Fig.3a. Current versus voltage function for diode is represented in Fig.3b. The capacitor with capacitance ***C*** is not charged initially. Then switch ***K*** is turned on.

What is the amount of heat that is produced in the resistor with resistance ***R*** while the capacitor is being charged?

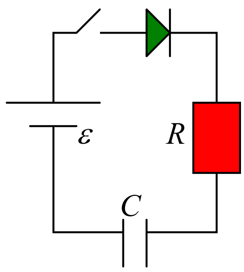

Fig.3a

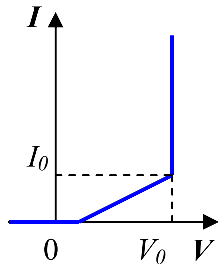

Fig.3b

Solution

An ideal diode characteristic would be as it is shown in Fig.3b-1. However, a typical diode characteristic is more like in Fig.3b-2. Notice that the diode conducts a small current in the forward direction up to a threshold voltage, e.g. 0.3 for germanium and 0.7 for silicon; after that it conducts as we might expect. The forward voltage drop is specified at a forward current.

To simplify a solution Fig.3b represents a broken line instead of a smooth curve for the real forward current vs forward voltage function. Besides, the value of threshold voltage becomes clearer.

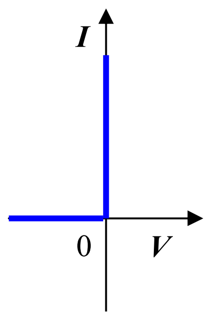

Fig.3b-1

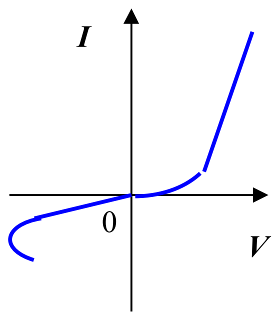

Fig.3b-2

In our problem after the circuit has been closed there appears a current which value is gradually decreasing as the capacitor is being charged, and its voltage is increasing. Unless the current becomes I_0 the voltage across diode continues to be V_0. During this initial time period in the resistor the heat H_1 is produced that can be calculated from the law of conservation of energy as a difference of total work (A) produced by a source, and a sum of works done to charge a diode (A_D), and a capacitor (A_C):

$$H_1 = A_1 - A_D - A_C \qquad (1)$$

At the moment $t=t_1$, when current becomes I_0, we can calculate the following values.

Voltage across capacitor:

$$V_C = \varepsilon - V_0 - I_0 R$$

Total charge passed through the circuit:

$$q_1 = CV_C = C(\varepsilon - V_0 - I_0 R)$$

Work produced by a source:

$$A_1 = q_1\varepsilon = C\varepsilon(\varepsilon - V_0 - I_0 R)$$

Work to charge a diode:

$$A_D = q_1 V_0 = CV_0(\varepsilon - V_0 - I_0 R)$$

Work to charge a capacitor:

$$A_C = \frac{q_1 V_C}{2} = \frac{C(\varepsilon - V_0 - I_0 R)^2}{2}$$

In Eq.(1) we can substitute unknowns by the calculated values and determine H_1:

$$H_1 = \frac{C}{2}\left[(\varepsilon - V_0)^2 - (I_0 R)^2\right] \qquad (2)$$

The further decreasing in current from I_0 to zero entails the linear decreasing of voltage across the diode (Fig.3b) as for a resistor that obeys the Ohm's law (we should neglect a short horizontal segment of the broken line). For our preliminary calculations it is convenient to symbolize the diode resistance as $r = V_0/I_0$, the heat evolved from the diode as H_{D2}, and the energy change in the capacitor as ΔW_C

Eq.(1) now transforms into the following:

$$H_2 = A_2 - H_{D2} - \Delta W_C \qquad (3)$$

At the moment when current becomes zero voltage across diode and the true resistor also becomes zero. Voltage across capacitor becomes ε with the charge of capacitor $q = \varepsilon C$.

Thus, during the second time interval $t_2 - t_1$ the charge $q_2 = q - q_1$ passes through the circuit.

$$q_2 = C\varepsilon - C(\varepsilon - V_0 - I_0 R) = C(V_0 + I_0 R)$$

Thus, during the second time interval a source produces work that equals

$$A_2 = q_2\varepsilon = C\varepsilon(V_0 + I_0 R) \qquad (4)$$

Change in the capacitor energy during this time interval equals to

$$\Delta W_C = \frac{q^2 - q_1^2}{2C} = \frac{C\varepsilon^2}{2} - \frac{C}{2}(\varepsilon - V_0 - I_0 R)^2 = C\varepsilon(V_0 + I_0 R) - \frac{C}{2}(V_0 + I_0 R)^2 \qquad (5)$$

Eqs.(3), (4), and (5) permit us to calculate the following

$$H_2 + H_{D2} = A_2 - \Delta W_C = \frac{C}{2}(V_0 + I_0 R)^2 \qquad (6)$$

Resistor and diode are connected in series. That is why the same current flows through them. In such case it is possible to write a following proportion:

$$\frac{H_2}{H_{D2}} = \frac{R}{r} \qquad (7)$$

Solving together Eqs.(6) and (7) with $r = V_0/I_0$, we can calculate H_2:

$$H_2 = \frac{R}{R + r}\frac{C(V_0 + I_0 R)^2}{2}$$

The resultant heat evolved by the resistance is given by

$$H_R = H_1 + H_2 = \frac{C\left[(\varepsilon - V_0)^2 - (I_0 R)^2\right]}{2} + \frac{R}{R + r}\frac{C(V_0 + I_0 R)^2}{2} = \frac{C}{2}\left[(\varepsilon - V_0)^2 + V_0 I_0 R\right]$$

4. Either in Guelph, or in CERN.

In the circular accelerator with radius ***R*** a very thin beam of protons with non-relativistic speed is injected. The mass ***m*** and the charge ***e*** of proton are well known. The initial current in the accelerator is ***I,*** the total number of particulates is ***n***. The magnetic flux through the beam circuit changes with the rate of φ, while the radius of the beam track remains unaltered.

What is the value of current after one turn of the particles?

What is the name of such kind accelerator?

Solution

To simplify the problem let us assume that the bunch of protons is injected and then circulate in the accelerator chamber as a sole particle with charge *ne*. Upon the definition a current is a rate at which electric charge flows through the cross section of conductor. If the motion of protons was uniform the relationship between the value of current I_U and speed v_U in the chamber could be expressed as:

$$I_U = \frac{en}{T} = \frac{env_U}{2\pi R}$$

Factor $\frac{en}{2\pi R}$ does not depend on time. That is why just the same formula can be used for the instantaneous current *I(t)* and speed *v(t)* of protons:

$$I(t) = \frac{en}{T} = \frac{env(t)}{2\pi R} \qquad (1)$$

To calculate current after one turn it is necessary to determine the speed of protons after one turn. It can be done using the law of conservation of energy and the law of electromagnetic induction. The magnitude of electromotive force of induction is equal to φ, its work produced to accelerate a proton is $e\varphi$. This work results in increasing of the kinetic energy of a proton. After one turn a speed of a proton can be calculated from the law of conservation of energy as:

$$v_1 = \sqrt{v_0^2 + \frac{2e\varphi}{m}} \qquad (2)$$

where v_0 is initial speed of a proton, and can be obtained from Eq.(1) with given initial current *I*:

$$v_0 = \frac{2\pi RI}{en} \qquad (3)$$

Combining Eqs.(1), (2), and (3) we can calculate current I_1 after one turn as:

$$I_1 = \frac{en}{2\pi R} v_1 = \frac{en}{2\pi R}\sqrt{\left(\frac{2\pi RI}{en}\right)^2 + \frac{2e\varphi}{m}} = I\sqrt{1 + \frac{n^2 e^3 \varphi}{2\pi^2 R^2 m I^2}}$$

5. Imagine your experiment.

Take a battery with known value of EMF ε (electromotive force) and a number ***N*** of equivalent capacitors with capacitance ***C*** and wires for connecting circuit.

What is the maximal value of the potential difference that you can obtain with this kit?

What will be the sequence of your actions to achieve this result?

Solution

It seams reasonable to begin an experiment with two capacitors *(N=2)*. After charging of both capacitors by the source of emf we can then connect them in series with the source, and obtain the potential difference of 3ε.

How can we exceed this value?

Let us connect the charged capacitors with the source in different way, as it is shown in Fig.5-1. After the connection has been fulfilled the charges of capacitors and the voltage across them change. The new values are respectively q_{11}, q_{21}, V_{11}, V_{21}. They are related with values C and ε as follows:

$$\varepsilon + V_{11}=V_{21} \qquad (1)$$

$$q_{11}+ q_{21}=2C\varepsilon \qquad (2)$$

$$q_{11} =CV_{11};\ q_{21} =CV_{21} \qquad (3)$$

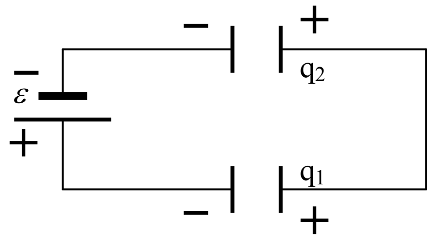

Fig.5-1

Solution of this system for voltage and charge gives

$$V_{21} = \frac{3}{2}\varepsilon \qquad (4)$$

$$q_{21}= \frac{3}{2}\varepsilon C$$

At the step 2 we should repeat charging of the first capacitor from the source of emf until it obtains the potential difference ε. After this let us again connect the same kind of circuit as in the Fig.5-1, and calculate the new value of V_{22} according to the same pattern. The second step of charging gives:

$$\varepsilon + V_{12} = V_{22}$$

$$q_{12} + q_{22} = \frac{5}{2}\varepsilon C$$

$$V_{22} = \frac{7}{4}\varepsilon \qquad (5)$$

The sequence of factors in V_2 calculation is following:

$$1;\ \frac{3}{2};\ \frac{7}{4};\ \frac{15}{8};\ \frac{31}{16};\ \ldots\ \frac{A_i}{B_i};\qquad A_i =2A_{i-1} +1;\quad B_i =2B_{i-1}\ .$$

Multiple repetition of the described procedure permits to gain the potential difference at the second capacitor achieving the value of 2ε. When we connect again two capacitors and a source in series, the total voltage of 4ε can be obtained. This is the maximal value for two capacitors.

For three capacitors the order of actions and operations is similar. We can charge the third capacitor to the potential difference 4ε, connect it in series with two others and the source, and obtain the total voltage of 8ε.

In general, for N capacitors the maximal potential difference is $2^N\varepsilon$.