2003-2004 Physics Olympiad Preparation Program

– University of Toronto –

Solution Set 6: AC Circuits, Particles, and General

1. Circuit.

The source of alternating current in the circuit shown in Fig.1-1 produces a signal represented in Fig.1-2. Values of $r, R, C, V_0, \tau,$ and T are known.

Voltage across the capacitor changes very slightly during one period. Calculate the voltage across the capacitor in a great number of periods.

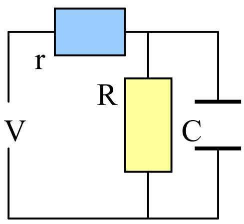

Fig.1-1

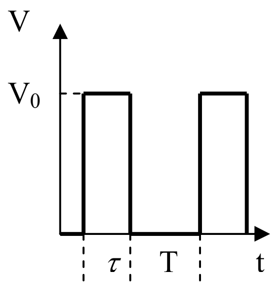

Fig.1-2.

<u>Solution</u>

When the circuit comes to equilibrium, voltage V_C and charge q_C of capacitor will not change any more. For the source of alternating current it means that the charge increase Δq in capacitor during the charging time interval τ will be compensated by the charge decrease during discharging. Apparently, we have a capacitor with big capacitance.

According to the problem situation, current through the circuit alters very slightly during the charging process, and can be calculated as an average value:

$$I_1 = \frac{\Delta q_0}{\tau}$$

Voltage across the capacitor during charging is given by:

$$V_C = I_R R = \frac{\Delta q_R}{\tau} R \quad \text{and} \quad V_C = V_0 - \frac{\Delta q_0}{\tau} r = V_0 - r\left(\frac{\Delta q_R}{\tau} + \frac{\Delta q}{\tau}\right) = V_0 - V_C \frac{r}{R} - r\frac{\Delta q}{\tau} \qquad (1)$$

If we assume that during discharging time T no current leaks through the source, charge from capacitor leaks through the second resistor with resistance R, with average current I_2:

$$I_2 = \frac{\Delta q}{T}$$

Voltage across capacitor during discharging is given by:

$$V_C = \frac{\Delta q}{T} R \qquad (2)$$

Combining Eqs.(1) and (2)we can calculate the unknown voltage, as follows:

$$V_C = V_0 \frac{R\tau}{R\tau + rT + r\tau}$$

2. Exotic atom.

In 1989 a new atom *protonium* was first produced. It consisted of proton and antiproton – the negatively charged particle with the proton mass and charge magnitude.

Calculate the energy of a photon, which is emitted as a result of the protonium transition from the state with the principle quantum number *n =2* into the state with *n =1*.

Solution

Calculation may be based on Bohr's model of the hydrogen atom. The Bohr's model describes the motion of electron classically, i.e. as a negatively charged point mass rotating around a proton. We can calculate the total energies of two states of the atom, and their difference will result in the energy of the emitted photon. According to the classical assumption, total energy may be obtained by addition of electric potential energy of the system of two charged point particles and their kinetic energies.

The main difference between protonium and hydrogen atom is the relative mass of two particles in the atom. In protonium both proton and antiproton with equal mass *m* can be considered to rotate around the center of mass of the atom at equal speed *v*. Center of mass is at the middle point between proton and antiproton, and the radius of revolution is *r*. The equation of motion for proton and antiproton is the same:

$$\frac{mv^2}{r} = \frac{ke^2}{(2r)^2} \qquad (1)$$

k is the electric constant ($9\text{x}10^9$ Nm^2/C^2), *e* is elementary charge.

The total kinetic energy for protonium is given by:

$$E_{kin} = 2 \cdot \frac{mv^2}{2} = \frac{ke^2}{4r} \qquad (2)$$

The potential energy of the protonium atom is given by:

$$E_{pot} = \frac{-ke^2}{2r} \qquad (3)$$

The total energy is the sum of the above values:

$$E = -\frac{ke^2}{4r} \qquad (4)$$

It is negative as any other potential energy of the system of attracting objects.

To calculate the radius of the orbit we need to apply the Bohr's assumption on the quantization of the total orbital angular momentum for the stable atom orbits:

$$2mvr = n\hbar, \qquad n = 1,2,3,..... \quad (5)$$

Combined solution of Eqs.(1) and (5) gives the quantized value of *r*, which depends on the integer number of specific stable orbit:

$$r_n = \frac{n^2\hbar^2}{kme^2}$$

As a result, quantization appears in the formula for energy. For our problem we can calculate the quantized energies for two states with *n=1,* and *n=2*. Their difference is the energy of a photon:

$$\Delta E_{21} = E_2 - E_1 = -\frac{k^2e^4m}{4\hbar^2}\left(\frac{1}{2^2} - 1\right) = \frac{3k^2e^4m}{16\hbar^2} = 9.36keV$$

3. **How the "hot" stars work.**

The carbon cycle of stars gives the following sequence of fuses and decays:

$$p + {}^{12}C \rightarrow {}^{13}N + \gamma$$
$${}^{13}N \rightarrow {}^{13}C + e^{+} + \nu$$
$$p + {}^{13}C \rightarrow {}^{14}N + \gamma$$
$$p + {}^{14}N \rightarrow {}^{15}O + \gamma$$
$${}^{15}O \rightarrow {}^{15}N + e^{+} + \nu$$
$$p + {}^{15}N \rightarrow {}^{12}C + {}^{4}He$$

γ is a $\gamma -$quantum; ν is a neutrino; e^+ is a positron.

Determine the energy, which is released during the formation of one mole of helium.

Solution

Situation of our problem describes so-called Bethe cycle, first proposed by Hans Bethe in 1939 for stars with the core temperature T >15x10^6 K. Such temperatures are sufficient to accelerate the main constituents of stars - hydrogen nuclei, or protons, which strike the carbon nuclei.

The total release of energy at the end of one cycle can be calculated as the total change in rest energy of all particles in the reactions. Attentive glance at the reactions of this cycle clarifies that protons involved in the cycle are never the products of reaction. Carbon isotope ^{12}C is input nucleus and also the output, and that is why we should not take it into account. Other isotopes undergo transmutations and are not the products of the cycle.

Thus, it is convenient to replace the real series of reactions by one, as follows:

$$4p \rightarrow {}^{4}He + 2e^{+} + 3\gamma + 2\nu$$

Energy released during the formation of one ^{4}He nucleus is given by:

$$E_1 = (4m_p - m_{He} - 2m_e)\, c^2$$

For one mole of helium this energy must be multiplied by the Avogadro's constant:

$$E = N_A c^2 \left(4m_p - m_{^4He} - 2m_{e^+}\right) =$$
$$= 6.0221 \cdot 10^{23} \cdot 8.9874 \cdot 10^{16} \cdot \left(4 \cdot 1.6726 \cdot 10^{-27} - 6.6439 \cdot 10^{-27} - 2 \cdot 9.1094 \cdot 10^{-31}\right) =$$
$$= 2.4181 \text{x} 10^{12}\ J = 2.4181\ TJ \text{ (terajoules)}$$

4. **Where are the lost neutrons?**

The beam of neutrons loses 15% of its initial number after passing through the cadmium plate with thickness 1 mm. The velocity of neutrons does not change. What fraction of neutrons passes through the cadmium plate with thickness 10 mm?

Solution

The fraction of lost neutrons after 1mm thickness layer of cadmium is:

$$a = N_L/N = 0.15$$

The number of transited neutrons is correspondingly:

$$N_1 = (1 - a)\, N$$

After the second 1mm-layer N_2 neutrons pass:

$$N_2 = (1 - a)\, N_1 = (1 - a)^2\, N$$

The fraction of neutrons passed through the 10 mm cadmium plate (so-called cadmium filter) is:

$$N_{10}/N = (1 - a)^{10} = 0.85^{10} = 0.2$$

5. **Experiment in the Department of Physics**

When you become a physicist at the Department of Physics UofT you will have an opportunity to fulfill the following experiment.

Fig.2 shows a picture of a particle track in the so-called Wilson's cloud chamber. The chamber was filled with the mixture of hydrogen vapor (H_2), alcohol vapor (C_2H_5OH), and water vapor (H_2O) and put into the magnetic field with induction of 1.3 T. The decay of vapor nuclei is caused by the interaction with speedy neutrons. Induction vector is perpendicular to the picture plane. ACA_1 – trajectory of a proton born at a point A.

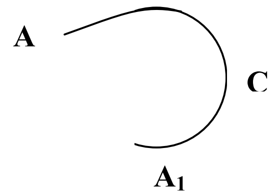

Fig.5-1

1) Propose a method to determine the proton energy at any point of its trajectory.
2) Why is the radius of curvature decreasing along the trajectory?
3) What nucleus experienced decay at the point A if we knew that A was a starting point on the tracks of two protons and two α-particles?

Solution

This was actually an experimental problem. To solve it completely you needed to sketch and work with Fig.5-1.

1) We know that the track belongs to proton with mass $m_p = 1.67x10^{-27}$ *kg*, and charge $e = 1.6x10^{-19}$ *C*, which moves in the magnetic field $\vec{B}$ with some velocity $\vec{v}$. Its velocity, induction of magnetic field, and radius *R* of curvature of the trajectory are combined into the equation of circular motion as follows:

$$\frac{m_p v^2}{R} = e\left|\vec{v}\right|\left|\vec{B}\right|$$

This equation permits to calculate an instantaneous velocity as a function of radius:

$$\left|\vec{v}\right| = \frac{e\left|\vec{B}\right|}{m_p} R$$

The proton kinetic energy is given by:

$$E_K = \frac{m_p v^2}{2} = \frac{\left(e\left|\vec{B}\right|\right)^2}{2m_p} R^2 \qquad (1)$$

Thus, Eq. (1) may be used to obtain the instantaneous kinetic energy graphically measuring only the radius of curvature *R*. The proposed method cannot be applied without the assumption on the way of radius measurement. Suppose we need to determine radius *R* at the point *M*. Let us choose two points *L* and *N* on the curve close to *M*. Then we should connect *L* and *M, M* and *N*, and graph a perpendicular at the middle points of segments *LM* and *MN*. The cross-point of the perpendiculars gives the center of curvature. Now we have only to measure the radius *R* = *OM*. Uncertainty in the radius measurement

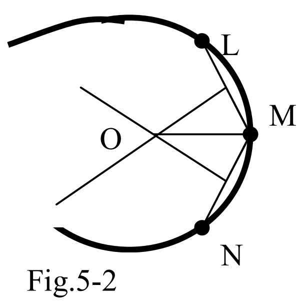

Fig.5-2

influences the accuracy of the obtained energy of proton in each point of curvature.
In the Fig. 5-1 radius of curvature at the point *C* is approximately 1 cm. If we assume that the figure represents a picture of real experiment, Eq. (1) gives the following kinetic energy of a proton at point *C*:

$$E_{KC} = \frac{\left(1.6 \cdot 10^{-19} \cdot 1.3\right)^2}{2 \cdot 1.67 \cdot 10^{-27}} \cdot 10^{-4} J = 1.30 \cdot 10^{-15} J = 8.10 keV$$

2) According to the Eq. (1) radius decreases due to the loss of kinetic energy. It happens because energetic protons ionize molecules of gas in the non-elastic collisions.
3) $\alpha -$particle is a nucleus of helium, or ${}_2^4He$. A proton is a nucleus of hydrogen, or ${}_1^1H$. The nuclear fission reaction of the unknown nucleus after interaction with neutron can be written as:

$$ {}_Z^AX + {}_0^1n \to 2{}_2^4He + 2{}_1^1H + k{}_0^1n \qquad (2)$$

At this step we are not sure that $k \neq 0$, but there may be neutrons in the product.
Law of conservation of charge gives Z=6. Hence, we know that the nucleus was ${}_6^{12}C$. We should choose the stable isotope for carbon nucleus.
From the law of conservation of mass:

$$k = 12 + 1 - 8 - 2 = 3.$$

The equated reaction is following:

$${}_6^{12}C + {}_0^1n \to 2{}_2^4He + 2{}_1^1H + 3{}_0^1n$$

Both protons and $\alpha -$particles are charged particles, and can be observed by their curved tracks in the chamber. Neutron is electrically neutral, and cannot produce ions in gas. That is why neutrons are not observed in the ionizing chambers.

6. Some comments on the Problem 4 (Accelerator) from the PS#5.

The main difficulty in this problem was connected with the discrepancy of the results of two different ways of calculation of the protons speed after one turn.
According to the law of conservation of energy and the law of electromagnetic induction one could obtain the value of speed, as it was calculated in the POPTOR solution:

$$v_1 = \sqrt{v_0^2 + \frac{2e\varphi}{m}} \qquad (6\text{-}1)$$

Eq. (6-1) was obtained without any information about the structure of magnetic field. The problem did not submit us with such information!

Another possible way to calculate a speed was to apply the second Newton's law to the circular motion of the proton and assumption on its uniform tangent acceleration *a*:

$$2\pi R = \frac{v_1^2 - v_0^2}{2a} \Longrightarrow v_1 = \sqrt{v_0^2 + 4\pi a R} \qquad (6\text{-}2)$$

$$evB = \frac{mv^2}{R} \Longrightarrow v = \frac{eRB}{m} \Longrightarrow a = \frac{dv}{dt} = \frac{eR}{m}\frac{dB}{dt} = \frac{eR\varphi_1}{m\pi R^2} = \frac{e\varphi_1}{\pi m R} \qquad (6\text{-}3)$$

After substitution of *a* in Eq. (6-2) we have for speed the following expression:

$$v_1 = \sqrt{v_0^2 + \frac{4e\varphi_1}{m}} \qquad (6\text{-}4)$$

Comparison of Eqs. (6-1) and (6-4) gives two different values for the magnetic flux rate:

$$\varphi = 2\varphi_1 \qquad (6\text{-}5)$$

This equation is known as a *betatron condition*. It shows that for the maintenance of the proton orbit with constant radius the given average flux rate of the orbit φ must be twice as big as the local flux rate φ_1 on the orbit.
General conclusion resulting from our solution is following: magnetic field in such kind of accelerator cannot be uniform when the radius of protons orbit is constant.

The betatron condition obtained in our solution does not mean that protons are accelerated in the betatrons. The betatron is the electron accelerator. Protons are accelerated in different kinds of cyclotrons (and linear accelerators also!).
The question of the problem was put only for 1 turn of the proton beam. It was done because protons experience the main acceleration once during the turn in the gap with strong electric field and then are accelerated and kept on their orbit by the growing non-uniform magnetic field. Such accelerator is called *synchrophasotron,* and we observed the protons on their stable orbit between two short intervals of electric acceleration.
No one participant was charged for the wrong name of accelerator.