2004-2005 Physics Olympiad Preparation Program

– University of Toronto –

Solutions. Set 2: Mechanics

Problem 1

Two little balls with masses m_1 and m_2 are connected with a spring and lie on the smooth horizontal surface. The spring constant is k. The balls are brought close to each other, and the spring becomes compressed. Then they are simultaneously released.
Determine the period of the resultant oscillations of two balls.
The period of oscillations of a point mass on a spring is given by:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Solution

Centre of mass of the system must be at rest. That is why two balls will oscillate with the opposite phases and equal periods. If friction is negligible, two spring forces originating from two displacements of the balls must be equal:

$$k_1x_1 = k_2x_2, \qquad (1)$$

where k_1 and k_2 are the spring constants for the two pieces of spring with the lengths l_1 and l_2; x_1 and x_2 are the values of corresponding displacements of two balls from their equilibrium positions. At the same time, l_1 and l_2 are the distances from the balls to the centre of mass:

$$l_1m_1 = l_2m_2 \quad \text{and} \quad l_1 + l_2 = l \qquad (2)$$

Solving the system of equations (2) we can find that:

$$l_1 = l\frac{m_2}{m_1 + m_2}; \ l_2 = l\frac{m_1}{m_1 + m_2} \qquad (3)$$

For any q, the elongation of the 1/q part of a spring is q times less than the elongation of the full spring, and the spring constant of the 1/q part is q times greater than the spring constant of the full spring. That is why:

$$k_1 = \frac{m_1 + m_2}{m_2}k$$

As $T_1 = T_2$, and $T_1 = 2\pi\sqrt{\frac{m_1}{k_1}}$, the period of oscillations of the system is:

$$T = 2\pi\sqrt{\frac{m_1m_2}{(m_1 + m_2)k}}$$

Problem 2

A rope is thrown over a pulley with its one part on the table of height h, and its another part on the floor, as it is shown in fig.1. After the rope is released it starts to move.
Find the speed of the steady motion of the rope.

Solution

The shortest solution uses the second Newton's law.
During the time interval Δt a new portion of rope with the length of $\Delta l = v\Delta t$ starts to move. Its mass is $\Delta m = \mu\Delta t = \mu v\Delta t$, where μ is the linear density of the rope. This mass has linear momentum $\Delta mv = \mu v^2\Delta t$, which is equal to the impulse of the net force of gravity of the left and right parts of the rope. The net force equals μgh. The second Newton's law gives us the following:

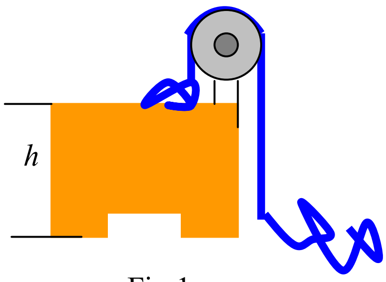

Fig.1

$$\mu v^2\Delta t = \mu gh\Delta t$$

From this equation, we can obtain the speed:

$$v = \sqrt{gh}$$

Problem 3

At the height of 200 km the density of the Earth's atmosphere is 1.6* 10^{-10} kg/m^3. A satellite has a mass of 10 kg and a cross sectional area of 0.5 m^2.
Estimate the resistance force experienced by the satellite at this altitude.

Solution

To solve the problem in general we need to introduce symbols for the given values, e.g.: h = 200 km, ρ = 1.6x10^{-10} kg/m^3, A = 0.5 m^2, m = 10 kg, v is the speed of a satellite.
The solution of the problem is based on the law of conservation of the linear momentum and the definition of force as a rate of change of the linear momentum.
During the time interval Δt a satellite meets the particles of ambient air that are contained in the volume $Av\Delta t$. The mass of these particles equals $\rho Av\Delta t$. During the collision with the satellite the particles obtain linear momentum $\rho Av\Delta t\, v = \rho Av^2\Delta t$. The rate of change of this linear momentum is the force exerted on the particles by the satellite. According to the 3rd Newton's law the force of the same magnitude but of the opposite direction is experienced by the satellite.

$$F = \rho Av^2 \qquad (1)$$

To find the force we have to know the speed of the satellite. For the circular motion along the orbit with radius $R = R_E + h$ (R_E is the radius of the Earth) the centripetal force is the force of gravity:

$$G\frac{Mm}{R^2} = \frac{mv^2}{R}$$

where G is the universal gravitational constant; M is the Earth's radius.

$$v^2 = G\frac{M}{R}$$

Now we will substitute v^2 in the formula (1) by the obtained expression:

$$F = \frac{\rho AGM}{R_E + h}$$

$G = 6.67 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$; $M = 5.98 \cdot 10^{24} kg$; $R_E = 6.37 \cdot 10^6 m$. The calculation gives us

$$F = 4.86 \text{ x } 10^{-3}\ N$$

Problem 4

Two identical little balls are connected with a string. One of them is thrown up with the initial speed *v*. What is the maximum altitude of the system?

Solution

We will consider the string non-stretched and weightless.

At the initial moment, the linear momentum of the system equals *mv*, while the mass of the system equals *2m*. That is why the initial speed of the centre of mass of the system equals

$$\frac{mv}{2m} = \frac{v}{2}$$

The law of conservation of energy for the system of two balls can be written for its centre of mass. The maximum altitude of the centre of mass *h* can be determined from the equation:

$$2mgh = \frac{2m\left(\frac{v}{2}\right)^2}{2}$$

$$h = \frac{v^2}{8g}$$

It must be mentioned that when the centre of mass of the system riches the apex of its trajectory *h,* two balls are not at rest. They move with the total kinetic energy

$$K = \frac{mv^2}{2} - 2mgh = \frac{mv^2}{2} - \frac{mv^2}{4} = \frac{mv^2}{4}.$$

Problem 5

Propose an experiment to determine the acceleration due to gravity and perform it with the obligatory use of the following equipment: ramp, tape measure, timer, and a roll of a bathroom tissue. Any other facilities can also be added if necessary.

Your result must contain the explanation and diagram of the experiment, the value of the obtained acceleration ***g***, the value of the error.

This problem could be solved in different ways. The POPTOR jury will select and post the best solutions of our students in the Comments to the Problem Set 2.

Our core idea of the solution was to apply the law of conservation of energy to the roll at the top and at the bottom of the ramp taking into consideration the rotation of a roll. In this experiment, it was necessary to measure the radius (or radii) of the roll, the height of the ramp, the length of the ramp, and the time elapsed to roll down the ramp. For the roll with the external radius R_2 and internal radius R_1 the moment of inertia equals $I = \frac{m(R_1^2 + R_2^2)}{2}$, and the law of conservation of energy can be written as:

$$mgh = \frac{Iv^2}{2R_2^2} + \frac{mv^2}{2}$$

where v is the speed of the roll at the bottom of the ramp.

$$g = \frac{v^2}{2h}\left(\frac{R_1^2 + R_2^2}{2R_2^2} + 1\right) \qquad (1)$$

If the roll is a solid cylinder, the equation is another ($R_1 = 0$):

$$g = \frac{3}{4}\frac{v^2}{h} \qquad (2)$$

The motion of the roll is uniformly accelerated without initial speed. That is why the length of the ramp L and the time t elapsed for the motion down the ramp relate as:

$$L = tv/2$$

It permits to find the speed as follows:

$$v = 2L/t$$

After the substitution v in the equation (1) or (2) we can calculate g.